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Three-dimensional analysis of the steady-state shape and small-amplitude oscillation of a bubble in uniform and non-uniform electric fields

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A three-dimensional analysis is performed to investigate the effects of an electric field on the steady deformation and small-amplitude oscillation of a bubble in dielectric liquid. To deal with a general class of electric fields, an electric field near the bubble is approximately represented by the sum of a uniform field and a linear field. Analytical results have been obtained for steady deformation and modification of oscillation frequency by using the domain perturbation method with the angular momentum operator approach.

It has been found that, to the first order, the steady shape of a bubble in an arbitrary electric field can be represented by a linear combination of a finite number of spherical harmonics Y_l^m, where $0 \leqslant l \leqslant 4$ and $|m| \leqslant l$. For the oscillation about the deformed steady shape, the overall frequency modification from the value of free oscillation about a spherical shape is obtained by considering two contributions separately: (i) that due to the deformed steady shape (indirect effect), and (ii) that due to the direct effect of an electric field. Both the direct and indirect effects of an electric field split the $(2l+1)$-fold degenerate frequency of Y_l^m modes, in the case of free oscillation about a spherical shape, into different frequencies that depend on m. However, when the average is taken over the $(2l+1)$ values of m, the frequency splitting due to the indirect effect via the deformed steady shape preserves the average value, while the splitting due to the direct effect of an electric field does not.

The oscillation characteristics of a bubble in a uniform electric field under the negligible compressibility assumption are compared with those of a conducting drop in a uniform electric field. For axisymmetric oscillation modes, deforming the steady shape into a prolate spheroid has the same effect of decreasing the oscillation frequency in both the drop and the bubble. However, the electric field has different effects on the oscillation about a spherical shape. The oscillation frequency increases with the increase of electric field in the case of a bubble, while it decreases in the case of a drop. This fundamental difference comes from the fact that the electric field outside the bubble exerts a suppressive surface force while the electric field outside the conducting drop exerts a pulling force on the surface.

1. Introduction

When a bubble or a drop is in an external field, the frequency of oscillation about the steady shape is changed from the natural frequency of free oscillation about spherical shape due to the effects of the external field. This frequency modification is of fundamental interest and it has been studied by many investigators. A few problems discussed in previous works include those of bubble oscillation in straining flow fields (Kang & Leal 1988), drop oscillation in electric fields (Brazier-Smith *et al.* 1971; Feng & Beard 1990, 1991; Kang 1993; Trinh, Holt & Thiessen 1996), and drop oscillation in magnetic fields (Suryanarayana & Bayazitoglu 1991; Cummings & Blackburn 1991; Bayazitoglu *et al.* 1996). The present work is also in the same category, and we are concerned with the effects of an electric field on the oscillation frequency of a bubble in a dielectric liquid. More specifically, we are interested in how the frequencies of axisymmetric and non-axisymmetric oscillations are changed under the influence of electric fields.

As examples of free-surface dynamics in electric fields, the problem of a conducting drop in free space and the problem of a bubble in a dielectric liquid are complementary to each other. In the conducting drop problem, the electric field is normal to the drop surface. On the other hand, in the problem of a bubble in a dielectric liquid, the electric field is nearly tangential to the bubble surface if the permittivity of the external dielectric fluid is much larger than that of the gas inside the bubble. Thus, the electric field is expected to have quite different effects on frequency modification in the two cases. In this sense, the two problems can be regarded as complementary to each other. For the problem of a conducting drop, Feng & Beard made significant contributions. They performed axisymmetric (Feng & Beard 1990) and three-dimensional (Feng & Beard 1991) analyses of the oscillation of a conducting drop in an electric field to determine the effect of a uniform electric field on the frequency change. However, the results for the complementary problem of bubble oscillation are not available yet. This fact provides a motivation of the present work.

In external fields, a bubble or a drop oscillates about the deformed non-spherical steady-state shape. Thus, in computation of oscillation frequency change, the deformed steady shape should be properly considered. This fact makes the problem quite difficult in many situations. However, in some problems, the overall frequency change due to the external field can be decomposed into two independent parts (to the accuracy of the first order of the external field): (i) the frequency change of oscillation about a spherical shape caused by the special effect of the external field, and (ii) the frequency change of free oscillation caused by a change in equilibrium shape. The above decomposition is possible if there is no fluid flow at steady state, as in the problem of a conducting drop in free space under an electric field. If the decomposition is possible, it is much more effective to discuss the two effects separately for better understanding of the oscillation characteristics in external fields. This has not been done yet except for the special case of axisymmetric oscillation of a drop (Kang 1993). Feng & Beard (1991) analysed the three-dimensional oscillation about the steady shape, but they considered the two effects together. On the other hand, a separate discussion is available for the problem of a drop in a magnetic field. Cummings & Blackburn (1991) obtained results for both the special magnetic field effect and the geometric effect of equilibrium shape change. In the present problem of a bubble in an electric field, the decomposition is also possible because there is no fluid flow at steady state under the assumption of no charge at the bubble surface. Thus, in this work, we consider the two effects separately.

A practical motivation of the present work can be found in the application to electrohydrodynamic boiling (Choi 1962; Bonjour, Verdier & Weil 1962) and bubble/drop dielectrophoresis (Jones & Bliss 1977; Pohl 1978; Feng 1996). In various processes, uniform and non-uniform electric fields are applied to modify the bubble dynamics. In particular, in the case of dielectrophoresis, quite strong non-uniform electric fields are applied to induce a change in bubble motion, or to position a bubble at a certain point inside a fluid medium which may be very useful in the study of bubble dynamics. Another related problem is the excitation of bubble oscillations with electric fields. Recently Bellini *et al.* (1997) performed an experiment in which weak time-periodic electric fields were used to selectively excite modes of different orders.

Theoretical results for bubble behaviour in non-uniform electric fields are very limited. Jones & Bliss (1977) studied the motion and the steady shape of a bubble in a non-uniform electric field. They computed the steady deformation of a bubble by using the spheroidal approximation. However, as will be shown later, in most situations, the bubble assumes non-spheroidal shapes in non-uniform electric fields. Thus, it is fair to say that theoretical results are not available even for the steady deformation. In the present work, we therefore want to develop theoretical results for steady deformation and oscillation of a bubble in non-uniform electric fields as well as in a uniform electric field. Furthermore, the results on the linear dynamical behaviour obtained in this work are expected to play an important role in the analyses of the nonlinear dynamical behaviour of a bubble under a time-periodic external field.

As preliminary steps for the analysis of oscillation, we obtain the electric field distribution inside and outside a spherical bubble in § 2, and the steady shapes in § 3. The electric field distribution of § 2 is used to predict the first-order steady deformation of a bubble under various types of non-uniform electric fields in § 3. In § 4, the effects of electric fields on the oscillation frequency modification are discussed in detail for the two types of electric fields: a uniform electric field and a straining electric field.

2. Electric force exerted on a spherical bubble

In this section, we consider the electric field distribution inside and outside a spherical bubble. To deal with a general class of electric fields, an electric field near the bubble is approximately represented by the sum of a uniform field and a linear field. Although only the results for a bubble in a perfect dielectric fluid under a static electric field will be the used in the further developments in §§ 3 and 4, we consider here a more general problem for later use in related works. We consider an alternating electric field and the leaky dielectric model of Melcher & Taylor (1969) is adopted with the assumption that the surface current is negligible. The same formulation was used by Torza, Cox and Mason (1971) for computing electrohydrodynamic deformation of liquid drops in a uniform electric field. The solution for a perfect dielectric is then obtained as a special case.

2.1. *Electric field distribution about a bubble*

We consider a spherical bubble of radius a located at $\boldsymbol{x}' = \boldsymbol{x}'_0$ in a dielectric fluid medium. The bubble is subject to a non-uniform electric field $\boldsymbol{E}(\boldsymbol{x}', t) = \hat{\boldsymbol{E}}(\boldsymbol{x}') \cos \omega t$ as shown in figure 1. It is assumed that the electrical properties are uniform in each of the outside and inside phases. Furthermore, it is assumed that there is no free charge in the fluid medium. The electrical permittivities of the phases outside and inside the bubble are denoted by ε and ε_{in}, and the conductivities by σ and σ_{in}.

Near the bubble, the time-independent part of the non-uniform electric field can be approximated by

$$\hat{\boldsymbol{E}}(\boldsymbol{x}') \simeq \hat{\boldsymbol{E}}(\boldsymbol{x}'_0) + (\boldsymbol{x}' - \boldsymbol{x}'_0) \cdot (\nabla \hat{\boldsymbol{E}}|_{x'_0}) \equiv \boldsymbol{E}_0 + \mathbf{G} \cdot x, \tag{2.1}$$

where $\boldsymbol{E}_0 = \hat{\boldsymbol{E}}(\boldsymbol{x}'_0)$, $\mathbf{G} = (\nabla \hat{\boldsymbol{E}}|_{x'_0})^T$, and $\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x}'_0$. In (2.1), $\mathbf{G}$ must be symmetric

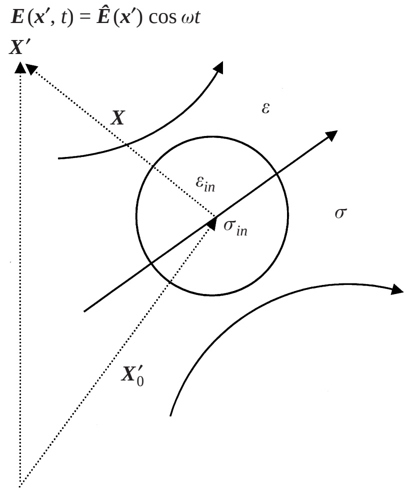

FIGURE 1. A bubble in a non-uniform electric field.

($\mathbf{G}^T = \mathbf{G}$) because $\hat{\boldsymbol{E}}$ is curl free, i.e.

$$\nabla \wedge \hat{\boldsymbol{E}} = \nabla \wedge (\mathbf{G} \cdot \boldsymbol{x}) = \varepsilon_{ijk} G_{ji} \boldsymbol{e}_k = 0. \tag{2.2}$$

Since $\varepsilon_{ijk} G_{ji} = 0$ for all k, $G_{lm} = G_{ml}$. From the condition of no free charge in the fluid medium, we have also the relation

$$\nabla \cdot \hat{\boldsymbol{E}} = \nabla \cdot (\mathbf{G} \cdot \boldsymbol{x}) = G_{ii} = 0. \tag{2.3}$$

For convenience, we adopt the notation $\boldsymbol{E}(\boldsymbol{x}, t) = \mathrm{Re}[\hat{\boldsymbol{E}}(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\omega t}]$ and $\boldsymbol{E}_{in}(\boldsymbol{x}, t) = \mathrm{Re}[\hat{\boldsymbol{E}}_{in}(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\omega t}]$, where Re denotes the real part of a complex number. We also introduce the electric potentials that satisfy $\hat{\boldsymbol{E}} = \nabla \Psi$ and $\hat{\boldsymbol{E}}_{in} = \nabla \hat{\Psi}_{in}$. Then the governing equations for the electric potentials are

$$\nabla^2 \hat{\Psi} = 0 \quad \text{and} \quad \nabla^2 \hat{\Psi}_{in} = 0. \tag{2.4}$$

As the boundary conditions, we have the far-field condition

$$\hat{\Psi} \to \hat{\Psi}_\infty = \boldsymbol{E}_0 \cdot \boldsymbol{x} + \tfrac{1}{2} \boldsymbol{x} \cdot \mathbf{G} \cdot \boldsymbol{x} \quad \text{as} \quad r \to \infty \tag{2.5}$$

and the matching conditions at the bubble surface

$$\varepsilon \boldsymbol{n} \cdot \nabla \hat{\Psi} \ - \varepsilon_{in} \boldsymbol{n} \cdot \nabla \hat{\Psi}_{in} = \hat{\sigma}_f, \tag{2.6}$$

$$\boldsymbol{t} \cdot \nabla \Psi \ - \boldsymbol{t} \cdot \nabla \Psi_{in} = 0, \tag{2.7}$$

$$\sigma \boldsymbol{n} \cdot \nabla \hat{\Psi} \ - \sigma_{in} \boldsymbol{n} \cdot \nabla \hat{\Psi}_{in} = -i\omega \hat{\sigma}_f, \tag{2.8}$$

where the surface free charge density is defined by $\sigma_f = \mathrm{Re}[\hat{\sigma}_f \ \mathrm{e}^{\mathrm{i}\omega t}]$, $r = \|\boldsymbol{x}\|$, $\boldsymbol{n}$ is the outwardly directed unit normal vector from the bubble surface, and $\boldsymbol{t}$ the unit tangent vector. Equation (2.6) is the jump condition of the normal component of the displacement vector, (2.7) represents the continuity of the tangential component of the electric field, and (2.8) is the unsteady balance of free charge at the bubble surface. In (2.8), the contribution of the surface current due to convection and conduction is neglected. For analysis, it is convenient to combine (2.6) and (2.8) to eliminate the surface free charge term as

$$\zeta \boldsymbol{n} \cdot \nabla \hat{\Psi} - \zeta_{in} \boldsymbol{n} \cdot \nabla \hat{\Psi}_{in} = 0 \quad \text{at} \quad r = a \tag{2.9}$$

where $\zeta = \sigma + \mathrm{i}\omega\varepsilon$.

Now, the solution procedure is straightforward. The external electric potential $\hat{\Psi}$ is given in terms of spherical harmonics as

$$\begin{aligned}\hat{\Psi} &= \hat{\Psi}_\infty + \hat{\Psi}' \\ &= (\boldsymbol{E}_0 \cdot \boldsymbol{x})\left(1 + \frac{A}{r^3}\right) + \tfrac{1}{2}(\boldsymbol{x} \cdot \mathbf{G} \cdot \boldsymbol{x})\left(1 + \frac{B}{r^5}\right)\end{aligned} \tag{2.10}$$

and the potential of the internal electric field is

$$\hat{\Psi}_{in} = C\boldsymbol{E}_0 \cdot \boldsymbol{x} + \tfrac{1}{2}D\ \boldsymbol{x} \cdot \mathbf{G} \cdot \boldsymbol{x}. \tag{2.11}$$

In (2.10) and (2.11), the coefficients A, B, C, D are obtained by using the matching conditions at the bubble surface ($r = a$). The outside and inside electric fields at the bubble surface ($\boldsymbol{x} = a\boldsymbol{n}$) are then given by

$$\hat{\boldsymbol{E}} = \left(1 + \frac{A}{a^3}\right)\boldsymbol{E}_0 - 3\frac{A}{a^3}(\boldsymbol{E}_0 \cdot \boldsymbol{n})\boldsymbol{n} + a\left(1 + \frac{B}{a^5}\right)(\mathbf{G} \cdot \boldsymbol{n}) - \frac{5}{2}\frac{B}{a^4}(\boldsymbol{n} \cdot \mathbf{G} \cdot \boldsymbol{n})\boldsymbol{n} \tag{2.12}$$

and

$$\hat{\boldsymbol{E}}_{in} = C\boldsymbol{E}_0 + D\mathbf{G} \cdot \boldsymbol{n}. \tag{2.13}$$

By substituting (2.12) and (2.13) into (2.7) and (2.9), we obtain

$$A = \frac{1-Z}{2+Z}a^3, \quad B = \frac{2(1-Z)}{3+2Z}a^5, \quad C = \frac{3}{2+Z}, \quad D = \frac{5}{3+2Z},$$

where $Z = \zeta_{in}/\zeta = (\sigma_{in} + \mathrm{i}\omega\varepsilon_{in})/(\sigma + \mathrm{i}\omega\varepsilon)$. Therefore, the electric fields at the bubble surface ($\boldsymbol{x} = a\boldsymbol{n}$) are given by

$$\boldsymbol{E} = A_1\boldsymbol{E}_0 - A_2(\boldsymbol{E}_0 \cdot \boldsymbol{n})\boldsymbol{n} + B_1\mathbf{G} \cdot \boldsymbol{n} - B_2(\boldsymbol{n} \cdot \mathbf{G} \cdot \boldsymbol{n})\boldsymbol{n}, \tag{2.14}$$

$$\boldsymbol{E}_{in} = A_1\boldsymbol{E}_0 + B_1\mathbf{G} \cdot \boldsymbol{n}, \tag{2.15}$$

where

$$A_1 = \mathrm{Re}\left[\frac{3\mathrm{e}^{\mathrm{i}\omega t}}{2+Z}\right], \quad A_2 = \mathrm{Re}\left[\frac{3(1-Z)\mathrm{e}^{\mathrm{i}\omega t}}{2+Z}\right],$$

$$B_1 = \mathrm{Re}\left[\frac{5a\mathrm{e}^{\mathrm{i}\omega t}}{3+2Z}\right], \quad B_2 = \mathrm{Re}\left[\frac{5a(1-Z)\mathrm{e}^{\mathrm{i}\omega t}}{3+2Z}\right],$$

with $Z = \zeta_{in}/\zeta = (\sigma_{in} + \mathrm{i}\omega\varepsilon_{in})/(\sigma + \mathrm{i}\omega\varepsilon)$. In general, Z is a complex number and the expressions for A_1, A_2, B_1, B_2 are very complicated. However, in the following special cases they have relatively simple expressions.

(i) Static electric field ($\omega = 0$): if the external fluid is not perfectly dielectric ($\sigma \neq 0$), Z reduces to $R \equiv \sigma_{in}/\sigma$ in the static case. Thus, we have

$$A_1 = \frac{3}{2+R}, \quad A_2 = \frac{3(1-R)}{2+R}, \quad B_1 = \frac{5a}{3+2R}, \quad B_2 = \frac{5a(1-R)}{3+2R}. \tag{2.16}$$

(ii) Perfect dielectric: if both phases are perfectly dielectric (i.e. $\sigma = \sigma_{in} = 0$), Z reduces to $q \equiv \varepsilon_{in}/\varepsilon$ in the oscillating field case ($\omega \neq 0$). Thus, we have

$$\left.\begin{aligned} A_1 &= \frac{3}{2+q}\cos\omega t, \quad A_2 = \frac{3(1-q)}{2+q}\cos\omega t, \\ B_1 &= \frac{5a}{3+2q}\cos\omega t, \quad B_2 = \frac{5a(1-q)}{3+2q}\cos\omega t. \end{aligned}\right\} \tag{2.17}$$

In the static case, we need to assume that $\hat{\sigma}_f = 0$ in order for (2.17) to be valid also for the limit $\omega = 0$.

(iii) Leaky dielectric with $R = \sigma_{in}/\sigma \to 0$ and $q = \varepsilon_{in}/\varepsilon \to 0$: in this case, $Z = 0$ and we have

$$A_1 = \tfrac{3}{2}\cos\omega t, \quad A_2 = \tfrac{3}{2}\cos\omega t, \quad B_1 = \tfrac{5}{3}a\cos\omega t, \quad B_2 = \tfrac{5}{3}a\cos\omega t. \tag{2.18}$$

Here we should mention the work of Feng (1996), who considered the special case of an axisymmetric static electric field. He obtained the electric field distribution and used his solution to study drop dielectrophoresis.

2.2. *Electric force exerted on a bubble*

Now let us compute the force exerted on the spherical bubble by the external electric field. The Maxwell stress tensor is given by

$$\boldsymbol{T}^e = \varepsilon(\boldsymbol{EE} - \tfrac{1}{2}E^2\boldsymbol{I}), \tag{2.19}$$

where $E^2 = \boldsymbol{E}\cdot\boldsymbol{E}$. Then the electric force per unit surface area is

$$\boldsymbol{f}^e \equiv \boldsymbol{n}\cdot\boldsymbol{T}^e = \varepsilon(E_n\boldsymbol{E} - \tfrac{1}{2}E^2\boldsymbol{n}), \tag{2.20}$$

where $E_n = \boldsymbol{n}\cdot\boldsymbol{E}$. Substituting (2.14) into (2.20), we have for the electric force exerted on unit surface area

$$\begin{aligned}\boldsymbol{f}^e = \varepsilon\big[&(A_1 - A_2)B_1(\boldsymbol{E}_0\cdot\boldsymbol{n})\mathsf{G}\cdot\boldsymbol{n} + A_1(B_1 - B_2)(\boldsymbol{n}\cdot\mathsf{G}\cdot\boldsymbol{n})\boldsymbol{E}_0\\ &-A_1B_1(\boldsymbol{E}_0\cdot\mathsf{G}\cdot\boldsymbol{n})\boldsymbol{n} + A_2B_2(\boldsymbol{E}_0\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\mathsf{G}\cdot\boldsymbol{n})\boldsymbol{n}\big]\\ &+\text{odd functions of } \boldsymbol{n}.\end{aligned} \tag{2.21}$$

The total electric force exerted on the spherical surface is then

$$\boldsymbol{F}^e = a^2\int_\Omega \boldsymbol{f}^e\,\mathrm{d}\Omega, \tag{2.22}$$

where Ω denotes the surface of a unit sphere. By using the fact that $\mathsf{G}^T = \mathsf{G}$ and the identities

$$\int_\Omega n_in_j\,\mathrm{d}\Omega = \frac{4\pi}{3}\delta_{ij}, \tag{2.23}$$

$$\int_\Omega n_in_jn_kn_l\,\mathrm{d}\Omega = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.24}$$

we can integrate (2.22) with f^e given in (2.21), to show that

$$\frac{\boldsymbol{F}^e}{\varepsilon a^2} = \frac{4\pi}{15}A_2(-5B_1 + 2B_2)\boldsymbol{E}_0\cdot\mathsf{G}. \tag{2.25}$$

For the special cases we have considered in (2.16), (2.17), (2.18), we have the following results:

(i) static electric field:

$$\boldsymbol{F}^e = -(4\pi\varepsilon a^3)\left(\frac{1-R}{2+R}\right)\boldsymbol{E}_0\cdot\mathsf{G}, \tag{2.26}$$

(ii) perfect dielectric:

$$\boldsymbol{F}^e = -(4\pi\varepsilon a^3)\left(\frac{1-q}{2+q}\right)\boldsymbol{E}_0\cdot\mathsf{G}\cos^2\omega t, \tag{2.27}$$

(iii) leaky dielectric in the limit $R = \sigma_{in}/\sigma \to 0$ and $q = \varepsilon_{in}/\varepsilon \to 0$:

$$\boldsymbol{F}^e = -(2\pi\varepsilon a^3)\boldsymbol{E}_0\cdot\mathsf{G}\cos^2\omega t. \tag{2.28}$$

Some comments should be made about the above special cases. In the case of perfect dielectric material, we have $(\boldsymbol{E}_0 \cos \omega t) \cdot (\mathbf{G} \cos \omega t) = \boldsymbol{E}(\mathbf{0}, t) \cdot (\nabla \boldsymbol{E}(\boldsymbol{x}, t)|_{\boldsymbol{x}=\mathbf{0}})^T = \frac{1}{2} \nabla |\boldsymbol{E}(\boldsymbol{x}, t)|^2$. Thus

$$\boldsymbol{F}^e = -(2\pi \varepsilon a^3) \left(\frac{1-q}{2+q} \right) \nabla |\boldsymbol{E}(\boldsymbol{x}, t)|^2. \tag{2.29}$$

This is the well-known result for a perfect dielectric sphere in another perfect dielectric medium (Pohl 1978). From (2.26)–(2.28), we can see also that the total electric force due to the given non-uniform electric field is proportional to a^3, and thus it is proportional to the bubble volume.

As mentioned in the introduction, hereinafter we limit our attention to the case of perfect dielectrics with zero surface free charge. As well known, if there is no surface free charge, electrical tangential stress is continuous across the interface and thus the hydrodynamic stress is also continuous. Therefore, there is no fluid flow induced by the electric field for a bubble with equilibrium shape at steady state. For more general leaky dielectric cases, we must consider also the induced fluid flow. The more general cases will be studied in future work.

3. Steady-state shape of a bubble in an electric field

In this section, we are concerned with the steady deformation of a bubble in a perfect dielectric liquid under a static non-uniform electric field. For simplicity, the net force exerted on the bubble is assumed to be zero. To do that, it is assumed that there is no gravity force if the electric field exerts no net force on the bubble or it is assumed that the gravity force cancels out the electric force if a net electric force is present. It is further assumed that the electric field is weak enough so that the electric field distribution obtained for a spherical bubble in § 2 may be used to obtain the first-order deformation.

The steady deformation of a bubble can be obtained by using the steady normal stress balance

$$[[T^e_{nn}]] + (P_{in} - P_\infty) = \gamma \nabla \cdot \boldsymbol{n}, \tag{3.1}$$

where $T^e_{nn} = \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \boldsymbol{T}^e)$ and $[[\ (\cdot)\ ]]$ denotes the outside quantity minus the inside quantity. Under the assumed conditions, (3.1) can be written by using (2.19) as

$$-\tfrac{1}{2}\varepsilon \left[(E^2 - qE^2_{in}) - 2\left((\boldsymbol{E} \cdot \boldsymbol{n})^2 - q(\boldsymbol{E}_{in} \cdot \boldsymbol{n})^2 \right) \right] + (P_{in} - P_\infty) = \gamma \nabla \cdot \boldsymbol{n}, \tag{3.2}$$

where $q = \varepsilon_{in}/\varepsilon$. If we non-dimensionalize the above equation by using the characteristic electric field scale E_c and the pressure scale $p_c = \gamma / a$, we have

$$-\tfrac{1}{2} W \Delta E^{*2} + (P^*_{in} - P^*_\infty) = \nabla^* \cdot \boldsymbol{n} \tag{3.3}$$

where W is the electrical Weber number defined by $W = (\varepsilon E_c^2 a)/\gamma$ and

$$\Delta E^{*2} = E^{*2} - qE^{*2}_{in} - 2\left[(\boldsymbol{E}^* \cdot \boldsymbol{n})^2 - q(\boldsymbol{E}^*_{in} \cdot \boldsymbol{n})^2 \right]. \tag{3.4}$$

A suitable characteristic scale E_c for the electric field will be determined later.

It might be useful to estimate the magnitude of W for a typical situation before starting our detailed analysis. For a bubble of radius of $a = 10^{-3}$ m in a dielectric liquid of $\varepsilon = 5 \times 10^{-11}$ F m^{-1} and $\gamma = 5 \times 10^{-2}$ N m^{-1}, we have $W = 10^{-2}$ for $E_c = 10^5$ V m^{-1}, and $W = 1$ for $E_c = 10^6$ V m^{-1}.

In this problem, we assume that the electric field is weak enough so that $0 < W \ll 1$. Then the first-order deformation can be obtained by using the solution for the electric field around a spherical bubble. We assume the bubble shape as

$$r^* = 1 + W\zeta_s(\theta,\phi) = 1 + W\sum_{l,m}\alpha_{lm}Y_l^m(\theta,\phi), \tag{3.5}$$

where $l \geqslant 0$ and $|m| \leqslant l$. Then the curvature term becomes

$$\nabla^*\cdot\boldsymbol{n} = 2 + W\sum_{l,m}\alpha_{lm}(l+2)(l-1)Y_l^m + O(W^2), \tag{3.6}$$

and the dimensionless internal pressure is given by

$$P_{in}^* = P_{g0}^*(1 - 3\kappa\alpha_{00}WY_0^0) + P_{vap}^* + O(W^2), \tag{3.7}$$

where P_{g0}^* is the inside gas pressure in the absence of an electric field and P_{vap}^* is the dimensionless vapour pressure. In (3.7), κ is the ratio of the specific heats for a gas bubble and $\kappa = 1$ for an isothermal vapour bubble. By substituting (3.6) and (3.7) into (3.3), we get

$$P_{g0}^* + P_{vap}^* - P_\infty^* = 2, \tag{3.8}$$

$$-\tfrac{1}{2}\Delta E^{*2} - 3\kappa\alpha_{00}P_{g0}^*Y_0^0 = \sum_{l,m}\alpha_{lm}(l+2)(l-1)Y_l^m. \tag{3.9}$$

If we expand ΔE^{*2} as

$$\Delta E^{*2} = \sum_{l,m}\langle l,m|\Delta E^{*2}\rangle Y_l^m, \tag{3.10}$$

where

$$\langle l,m|\Delta E^{*2}\rangle = \int_0^{2\pi}\int_0^{\pi}(\Delta E^{*2})Y_l^{m*}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{3.11}$$

and Y_l^{m*} is the complex conjugate of Y_l^m, then we have

$$\alpha_{00} = -\frac{\langle 0,0,|\Delta E^{*2}\rangle}{2(3\kappa P_{g0}^* - 2)}, \qquad \alpha_{lm} = -\frac{\langle l,m|\Delta E^{*2}\rangle}{2(l+2)(l-1)} \quad (l \neq 1). \tag{3.12}$$

If $\langle l,m|\Delta E^{*2}\rangle \neq 0$ for any m with $l = 1$, it means there is a net electric force exerted on the bubble. In that case, the net electric force must be cancelled out by the gravity force or others in order for the bubble to stay at the fixed position.

In the case of perfect dielectrics ($\sigma_{in} = \sigma = 0$), the static electric fields inside and outside the bubble are given by

$$\boldsymbol{E} = \left(\frac{3}{2+q}\right)[\boldsymbol{E}_0 - (1-q)(\boldsymbol{E}_0\cdot\boldsymbol{n})\boldsymbol{n}] + \left(\frac{5a}{3+2q}\right)[\mathbf{G}\cdot\boldsymbol{n} - (1-q)(\boldsymbol{n}\cdot\mathbf{G}\cdot\boldsymbol{n})\boldsymbol{n}], \tag{3.13}$$

$$\boldsymbol{E}_{in} = \left(\frac{3}{2+q}\right)\boldsymbol{E}_0 + \left(\frac{5a}{3+2q}\right)\mathbf{G}\cdot\boldsymbol{n} \tag{3.14}$$

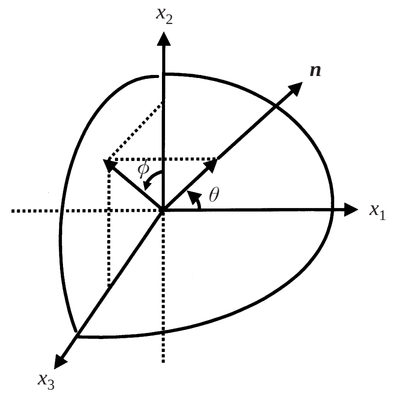

FIGURE 2. The coordinate system adopted in the present study.

from (2.14) and (2.15), and in the non-dimensional form they are given by

$$\boldsymbol{E}^* = \left(\frac{3}{2+q}\right)[\boldsymbol{E}_0^* - (1-q)(\boldsymbol{E}_0^*\cdot\boldsymbol{n})\boldsymbol{n}] + \left(\frac{5}{3+2q}\right)\left(\frac{aG}{E_c}\right)[\mathbf{G}^*\cdot\boldsymbol{n} - (1-q)(\boldsymbol{n}\cdot\mathbf{G}^*\cdot\boldsymbol{n})\boldsymbol{n}], \tag{3.15}$$

$$\boldsymbol{E}_{in}^* = \left(\frac{3}{2+q}\right)\boldsymbol{E}_0^* + \left(\frac{5}{3+2q}\right)\left(\frac{aG}{E_c}\right)\mathbf{G}^*\cdot\boldsymbol{n} \tag{3.16}$$

where G is the characteristic value of $\mathbf{G}$. Thus,

$$\Delta E^{*2} = \frac{9(1-q)}{(2+q)^2}[E_0^{*2} - (1-q)(\boldsymbol{E}_0^*\cdot\boldsymbol{n})^2] + \frac{30(1-q)}{(2+q)(3+2q)}\left(\frac{aG}{E_c}\right)[\boldsymbol{E}_0^*\cdot\mathbf{G}^*\cdot\boldsymbol{n} - (1-q)(\boldsymbol{E}_0^*\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\mathbf{G}^*\cdot\boldsymbol{n})] + \frac{25(1-q)}{(3+2q)^2}\left(\frac{aG}{E_c}\right)^2[(\boldsymbol{n}\cdot\mathbf{G}^{*2}\cdot\boldsymbol{n}) - (1-q)(\boldsymbol{n}\cdot\mathbf{G}^*\cdot\boldsymbol{n})^2]. \tag{3.17}$$

For convenience, we consider the coordinate system shown in figure 2, where the unit normal vector is given by

$$\boldsymbol{n} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi). \tag{3.18}$$

In the following subsections, we will consider several specific cases. However, at this point we can make a comment on the steady deformation of a bubble. As we can see in (3.17), the uniform field part (the first line of (3.17)) results in the steady deformation of $Y_0^0(\theta,\phi)$- and $Y_2^0(\theta,\phi)$-modes and the straining electric field part (the third line of (3.17)) results in the modes of $Y_0^0(\theta,\phi)$, $Y_2^m(\theta,\phi)$, $Y_4^m(\theta,\phi)$. The interaction terms of uniform field and the straining field result in the modes of $Y_1^m(\theta,\phi)$ and $Y_3^m(\theta,\phi)$. This general statement is valid for all cases if the bubble is sufficiently small so that the non-uniform field is approximated by

$$\boldsymbol{E}_\infty(\boldsymbol{x}) = \boldsymbol{E}_0 + \mathbf{G}\cdot\boldsymbol{x}. \tag{3.19}$$

3.1. *Uniform electric field*

For the uniform electric field of which the undisturbed electric field is given by $\boldsymbol{E}_\infty(\boldsymbol{x}) = E_0\boldsymbol{e}_1$, we take $E_c = E_0$. Then we have

$$\Delta E^{*2} = \frac{9(1-q)}{(2+q)^2}[1-(1-q)\cos^2\theta] = -\frac{3(1-q)}{(2+q)^2}P_2(\cos\theta) + \frac{3(1-q)}{2+q}P_0(\cos\theta). \tag{3.20}$$

In this case, the equilibrium bubble shape is given by $r^* = 1 + W\zeta_s(\theta,\phi)$ with

$$\zeta_s(\theta,\phi) = \frac{3(1-q)^2}{4(2+q)^2}P_2(\cos\theta) - \frac{3(1-q)}{2(2+q)(3\kappa P^*_{g0}-2)}P_0(\cos\theta), \tag{3.21}$$

where $P^*_{g0} = 2 + P^*_\infty - P^*_{vap}$, and $W = (\varepsilon E_0^2 a)/\gamma$. For the special case of $q = \varepsilon_{in}/\varepsilon \to 0$, we have

$$\zeta_s(\theta,\phi) = \tfrac{3}{16}P_2(\cos\theta) - \frac{3}{4(3\kappa P^*_{g0}-2)}P_0(\cos\theta). \tag{3.22}$$

Equation (3.22) indicates a quite interesting result. The steady bubble volume may be increased or decreased depending on the value of P^*_{g0}. If $P^*_{g0} > (2/3\kappa)$, the contribution of gas pressure is dominant inside the bubble and the bubble volume is decrease due to the compressive electric force. However, if $P^*_{g0} < (2/3\kappa)$, the bubble volume is increased. In this case, the contribution of the vapour pressure is dominant and the total pressure inside the bubble is not much affected by the volume change. When P^*_∞ is fixed, under the additional compressive electric force, the inside pressure can be kept nearly constant only by reducing the compressive force due to surface tension. Thus, the bubble radius should be increased as predicted in (3.22).

3.2. *Axisymmetric straining electric field*

In the case $\boldsymbol{E}_\infty(\boldsymbol{x}) = \mathbf{G}\cdot\boldsymbol{x}$, we take $E_c = aG$. Then the dimensionless electric field gradient of the straining electric field is given by

$$\mathbf{G}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{3.23}$$

and we have

$$\Delta E^{*2} = -\frac{90(1-q)^2}{7(3+2q)^2}P_4(\cos\theta) + \frac{25(1-q)(3+4q)}{14(3+2q)^2}P_2(\cos\theta) + \frac{5(1-q)}{2(3+2q)}P_0(\cos\theta). \tag{3.24}$$

The equilibrium shape is given by $r^* = 1 + W\zeta_s(\theta,\phi)$ with

$$\zeta_s(\theta,\phi) = \frac{5(1-q)^2}{14(3+2q)^2}P_4(\cos\theta) - \frac{25(1-q)(3+4q)}{112(3+2q)^2}P_2(\cos\theta) - \frac{5(1-q)}{4(3+2q)(3\kappa P^*_{g0}-2)}, \tag{3.25}$$

where $W = (\varepsilon G^2 a^3)/\gamma$. In the case of $q = \varepsilon_{in}/\varepsilon \to 0$, we have

$$\zeta_s(\theta,\phi) = \tfrac{5}{126}P_4(\cos\theta) - \tfrac{25}{336}P_2(\cos\theta) - \frac{5}{12(3\kappa P^*_{g0}-2)}. \tag{3.26}$$

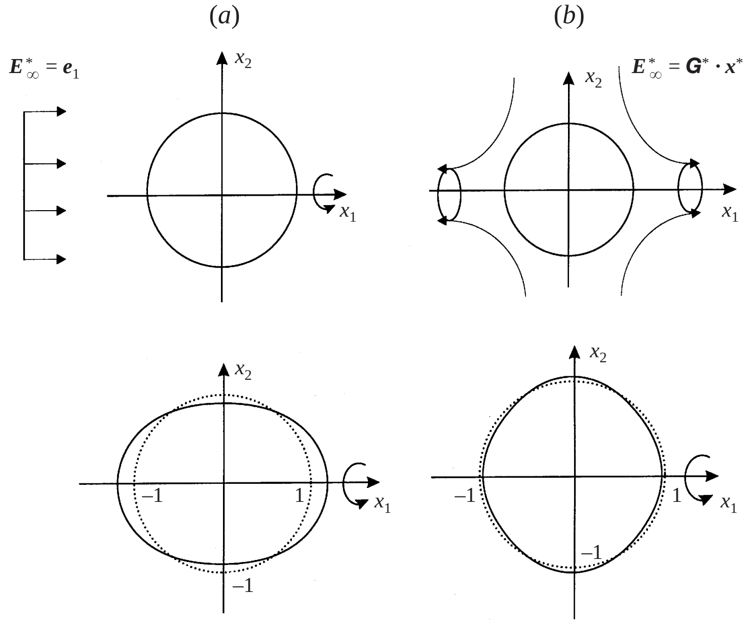

FIGURE 3. The equilibrium shapes of a bubble in a uniform field (*a*) and in an axisymmetric straining field (*b*).

The equilibrium shape of a bubble in an axisymmetric straining field is shown with that in a uniform field in figure 3. The parameter values are $W = 1$ and $P^*_{g0} = \infty$ (since the bubble deformations for small Weber numbers are very small, here we use intentionally a large value of W to show the deformation characteristics more clearly). As we can see in the figure, the bubble does not have a spheroidal shape in a straining electric field. It has the largest radius at the equator.

Thus far, we have considered axisymmetric electric fields. In the following, we consider several cases where non-axisymmetric equilibrium shapes are obtained. However, we consider only the case of $q \to 0$ for simplicity.

3.3. *Two-dimensional hyperbolic electric field*

In the case of two-dimensional hyperbolic electric field shown in figure 4(*a*), we have the dimensionless electric field gradient

$$\boldsymbol{G}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.27}$$

If $q = 0$, we have from (3.17)

$$\Delta E^{*2} = \tfrac{25}{9}\left[(\boldsymbol{n}\cdot\boldsymbol{G}^{*2}\cdot\boldsymbol{n}) - (\boldsymbol{n}\cdot\boldsymbol{G}^*\cdot\boldsymbol{n})^2\right] \tag{3.28}$$

and

$$\Delta E^{*2} = \tfrac{25}{9}\left[\cos^2\theta + (1-\cos^2\theta)\cos^2\phi \right.$$
$$\left. -\left(\cos^4\theta + (1-\cos^2\theta)^2\cos^4\phi - 2\cos^2\theta(1-\cos^2\theta)\cos^2\phi\right)\right]. \tag{3.29}$$

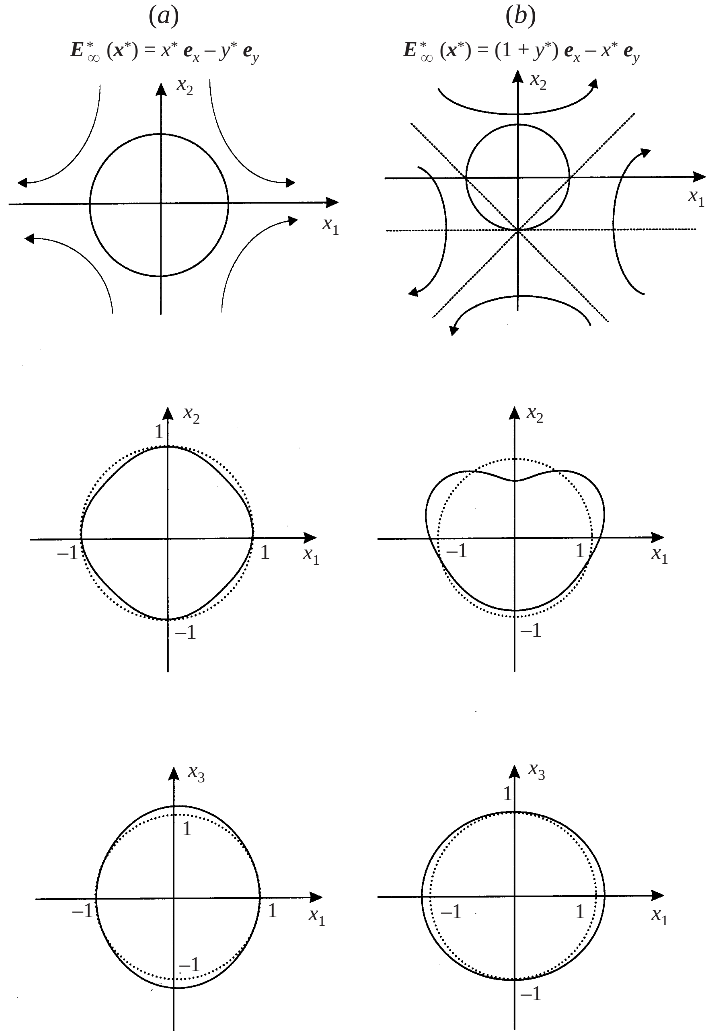

FIGURE 4. The equilibrium bubble shapes in a hyperbolic field (a) and in a combined electric field (b).

The equilibrium bubble shape is then given by $r^* = 1 + W\zeta_s(\theta, \phi)$ with

$$\zeta_s(\theta, \phi) = \sum_{l,m} \alpha_{lm} Y_l^m(\theta, \phi), \tag{3.30}$$

where $W = (\varepsilon G^2 a^3)/\gamma$ and

$$\alpha_{0,0} = -\frac{10(\pi)^{1/2}}{9(3P^*_{g0} - 2)}$$

$$\alpha_{2,0} = -\frac{5(5\pi)^{1/2}}{252}, \quad \alpha_{2,\pm 2} = -\frac{5(30\pi)^{1/2}}{504},$$

$$\alpha_{4,0} = \frac{95(\pi)^{1/2}}{3402}, \quad \alpha_{4,\pm 2} = -\frac{25(\pi/10)^{1/2}}{567}, \quad \alpha_{4,\pm 4} = \frac{25(\pi/70)^{1/2}}{486}$$

all other $\alpha_{l,m} = 0$.

The static shape of an incompressible bubble in the above hyperbolic electric field is shown in figure 4(*a*) for the electrical Weber number $W = 1$. As in the case of axisymmetric deformation, we use intentionally a large value of W to show the deformation characteristics clearly. If we look at the x_1, x_2 section at $x_3 = 0$ ($\phi = 0$ for $x_2 > 0$ and $\phi = \pi$ for $x_2 < 0$), we can see that the bubble is most deformed near $\theta = \pi/4$ and $\theta = 3\pi/4$ due to the strong electric field near those points. However, if we look at the x_1, x_3 section, we can see that the bubble is extended in the x_3-direction because the electric field is relatively weaker at the intersections of x_3-axis and the bubble surface than at the intersection of x_1-axis. Since a constant bubble volume is assumed, the bubble is extended to make up for the inward deformation at the x_1, x_2 section.

3.4. *Combination of uniform and hyperbolic fields*

As a special case where non-zero net electric force results, we consider an electric field

$$E_\infty(\boldsymbol{x}) = E_0\boldsymbol{e}_x + G(y\boldsymbol{e}_x + x\boldsymbol{e}_y) \quad \text{with} \quad G = \frac{E_0}{a}. \tag{3.31}$$

The dimensionless electric field is represented by

$$\boldsymbol{E}_0^* = (1,0,0) \quad \text{and} \quad \mathbf{G}^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.32}$$

and the corresponding electric field is shown in figure 4(*b*). Then from (3.17) we have for the $q = 0$ case

$$\begin{aligned}\Delta E^{*2} = {} & \tfrac{9}{4}(1-\cos^2\theta) + 5\sin\theta(1-2\cos^2\theta)\cos\phi \\ & +\tfrac{25}{9}(\cos^2\theta + \sin^2\theta\cos^2\phi - 4\cos^2\theta\sin^2\theta\cos^2\phi).\end{aligned} \tag{3.33}$$

In this example, $\langle 1, m|\Delta E^{*2}\rangle$ are not zero for all m. Specifically we have

$$\langle 1,1|\Delta E^{*2}\rangle = -(6\pi)^{1/2} \quad \text{and} \quad \langle 1,-1|\Delta E^{*2}\rangle = (6\pi)^{1/2}. \tag{3.34}$$

Now we show that (3.34) corresponds to non-zero net electric force dimensionless net electric force can be computed by

$$\begin{aligned}(\boldsymbol{F}^e)^* &= -\frac{1}{2}\int_0^{2\pi}\int_0^{\pi} \Delta E^{*2}\boldsymbol{n}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -\frac{1}{2}\int_0^{2\pi}\int_0^{\pi} \Delta E^{*2}(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\end{aligned} \tag{3.35}$$

Since

$$\sin\theta\cos\phi = -(2\pi/3)^{1/2}(Y_1^1 - Y_1^{-1}) = (2\pi/3)^{1/2}(Y_1^{-1^*} - Y_1^{1^*}), \tag{3.36}$$

we have

$$(F_2^e)^* = -\tfrac{1}{2}\left(2\pi/3\right)^{1/2}\left(\langle 1,-1|\Delta E^{*2}\rangle - \langle 1,1|\Delta E^{*2}\rangle\right) = -2\pi. \tag{3.37}$$

In dimensional form $F_2^e = -(2\pi\varepsilon a^3)E_0 G$, and this result is the same as that in § 2. The equilibrium shape is given by $r^* = 1 + W\zeta_s(\theta,\phi)$ with

$$\zeta_s(\theta,\phi) = \sum_{l,m}\alpha_{lm}Y_l^m(\theta,\phi), \tag{3.38}$$

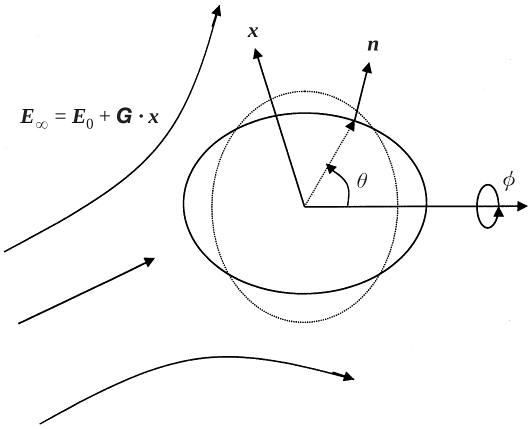

FIGURE 5. A bubble oscillating in a non-uniform electric field.

where $W = (\varepsilon G^2 a^3)/\gamma$ and

$$\alpha_{0,0} = -\frac{47(\pi)^{1/2}}{18(3\kappa P_{g0}^* - 2)}$$

$$\alpha_{2,0} = \frac{139(\pi/5)^{1/2}}{504}, \quad \alpha_{2,\pm 2} = -\frac{5(5\pi/6)^{1/2}}{84},$$

$$\alpha_{3,\pm 1} = \mp\frac{2(\pi/21)^{1/2}}{5},$$

$$\alpha_{4,0} = -\frac{40(\pi)^{1/2}}{1701}, \quad \alpha_{4,\pm 2} = \frac{100(\pi/10)^{1/2}}{1701} \tag{3.39}$$

all other $\alpha_{l,m} = 0$.

The static shape of an incompressible bubble in the above combined electric field is shown in figure 4(*b*) for $W = 1$. If we look at the x_1, x_2 section, we can see that the bubble is deformed into a heart shape. As shown in figure 4(*b*), the electric field is strongest near the point $(0, 1, 0)$ on the bubble surface while it becomes zero at the points in the middle of the first and second quadrants. Thus, the bubble is deformed inward near the point $(0, 1, 0)$, and deformed outward near the points of zero electric field.

4. Small-amplitude oscillation of a bubble

We consider now the small-amplitude oscillation of a bubble in a perfect dielectric fluid about the steady shape under a non-uniform electric field as shown in figure 5. The viscosity effect is assumed to be negligibly small so that the potential flow solution is the leading-order solution for the flow field. As non-axisymmetric steady shapes, we consider the cases of $q = \varepsilon_{in}/\varepsilon \to 0$ for simplicity. In this situation, the internal electric field does not contribute to the normal stress balance as shown below equation (3.1). Thus, we need to consider only the electric and velocity fields outside the bubble, and we use the electric and velocity potentials such that $\boldsymbol{E} = \nabla \Psi$ and $\boldsymbol{u} = \nabla \Phi$ in the problem formulation. The governing equations and the boundary conditions are non-dimensionalized by the following characteristic scales:

$$l_c = a, \quad t_c = \left(\frac{\rho a^3}{\gamma}\right)^{1/2}, \quad \Phi_c = \left(\frac{\gamma a}{\rho}\right)^{1/2}, \quad p_c = \frac{\gamma}{a}, \quad E_c. \tag{4.1}$$

The characteristic scale for the electric field E_c has not been specified here yet, and will be determined later. For convenience, hereinafter, the superscript asterisk for the dimensionless variables is dropped if not confusing. The dimensionless governing equation for the electric field is

$$\nabla^2 \Psi = 0, \tag{4.2}$$

and that for the fluid motion is

$$\nabla^2 \Phi = 0. \tag{4.3}$$

For the boundary conditions, we have the far-field conditions

$$\nabla \Phi \to 0, \quad \nabla \Psi \to \boldsymbol{E}_0 + \mathbf{G} \cdot \boldsymbol{x} \quad \text{as} \quad r \to \infty, \tag{4.4}$$

and the conditions at the bubble surface:

(i) kinematic condition

$$-\frac{\partial F}{\partial t} = \nabla \Phi \cdot \nabla F, \tag{4.5}$$

(ii) normal stress condition

$$-\tfrac{1}{2} W E^2 + \frac{\partial \Phi}{\partial t} + \tfrac{1}{2} \nabla \Phi \cdot \nabla \Phi + 2S \frac{\partial^2 \Phi}{\partial n^2} - p_v + (P_{in} - P_\infty) = \nabla \cdot \boldsymbol{n}, \tag{4.6}$$

(iii) zero permittivity ratio condition ($q = \varepsilon_{in}/\varepsilon \to 0$)

$$E_n = \boldsymbol{n} \cdot \nabla \Psi = 0, \tag{4.7}$$

where $F = r - R(\theta, \phi, t)$ is the shape function for the bubble surface. The term $2S(\partial^2 \Phi / \partial n^2)$ represents the viscous normal stress and p_v is the pressure correction due to the weak viscous effect. The dimensionless number S is defined as (for a more detailed discussion of these terms, see Kang & Leal 1988)

$$S = \frac{\mu}{(\rho a \gamma)^{1/2}} = \frac{t_c}{a^2/(\mu/\rho)}. \tag{4.8}$$

In the present problem, we again consider the case $0 < W \ll 1$. Then the small-amplitude oscillation problem can be effectively analysed by the domain perturbation method. To do that we assume that the potentials and the shape function are expanded to $O(\varepsilon)$, where ε is the small parameter for small-amplitude oscillation, as

$$\Psi = \Psi_s + \varepsilon \Psi_u, \quad \Phi = \varepsilon \Phi_u, \quad R = 1 + W \zeta_s + \varepsilon \zeta_u, \tag{4.9}$$

where the subscript u denotes the unsteady state. For our analysis of the non-axisymmetric shape oscillation, we introduce the angular momentum operator which is defined by

$$\mathbf{L} = -\mathrm{i} \boldsymbol{r} \wedge \nabla, \tag{4.10}$$

where $\boldsymbol{r}$ is the position vector. Then we use identities

$$\nabla f \cdot \nabla g = \frac{\partial f}{\partial r} \frac{\partial g}{\partial r} - \frac{1}{r^2} (\mathbf{L} f \cdot \mathbf{L} g) \tag{4.11}$$

and

$$2(\mathbf{L} f \cdot \mathbf{L} g) = \mathrm{L}^2 (f g) - f \mathrm{L}^2 g - g \mathrm{L}^2 f, \tag{4.12}$$

where $L^2 = \mathbf{L}\cdot\mathbf{L}$ (see Arfken 1985 for the properties of the angular momentum operator). We can now show that the boundary conditions at the bubble surface can be transformed to the equivalent conditions at $r = 1$, which are accurate to $O(W)$, as the following.

(i) Kinematic condition

$$\frac{\partial \zeta_u}{\partial t} = \frac{\partial \Phi_u}{\partial r} + W\zeta_s \frac{\partial^2 \Phi_u}{\partial r^2} + \tfrac{1}{2}W\left[L^2(\Phi_u\zeta_s) - \Phi_u L^2(\zeta_s) - \zeta_s L^2(\Phi_u)\right]. \tag{4.13}$$

(ii) Normal stress condition

$$\begin{aligned} &\tfrac{1}{2}W\left[L^2(\Psi_s\Psi_u) - \Psi_s L^2(\Psi_u) - \Psi_u L^2(\Psi_s) - \zeta_u \frac{\partial}{\partial r}(\nabla\Psi_s\cdot\nabla\Psi_s)\right] \\ &\quad + \frac{\partial \Phi_u}{\partial t} + W\zeta_s\frac{\partial^2\Phi_u}{\partial r \partial t} + 2S\frac{\partial^2 \Phi_u}{\partial r^2} - \tilde{p}_v - 3\kappa\langle\zeta_u\rangle P_{g0} \\ &\quad = \left[-2\zeta_u + L^2(\zeta_u)\right] + 2W\left[2\zeta_s\zeta_u - \zeta_s L^2(\zeta_u) - \zeta_u L^2(\zeta_s)\right] \end{aligned} \tag{4.14}$$

where $\partial\Psi_s/\partial r = 0$ and

$$P_{in} = P_{g0}\langle(1 + W\zeta_s + \varepsilon\zeta_u)^3\rangle^{-\kappa} + P_{vap}. \tag{4.15}$$

For a gas bubble, we assume $\kappa = C_p/C_v$ and $P_{vap} = 0$, and for an isothermal vapour bubble, $\kappa = 1$ and $P_{vap} =$ const. In (4.14), $\langle(\cdot)\rangle$ is defined as

$$\langle(\cdot)\rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\cdot)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and $p_v = \varepsilon\tilde{p}_v$ is assumed because the viscous pressure correction is at most $O(\varepsilon)$.

(iii) Zero permittivity ratio condition

$$\frac{\partial \Psi_u}{\partial r} + \zeta_u\frac{\partial^2\Psi_s}{\partial r^2} + \tfrac{1}{2}\left[L^2(\zeta_u\Psi_s) - \zeta_u L^2(\Psi_s) - \Psi_s L^2(\zeta_u)\right] = 0. \tag{4.16}$$

As we can see in (4.13)–(4.16), the behaviour of small-amplitude oscillation is affected by (i) the steady deformation (represented by ζ_s), (ii) the electric field (represented by Ψ_s and Ψ_u), and (iii) the viscous effect (represented by the dimensionless number S and the viscous pressure correction term $\tilde{p}_v$). Up to $O(W)$ or $O(S)$, these three effects are independent of each other and can be treated separately for simplicity.

4.1. *The effects of axisymmetric steady deformation*

In this section, we do not deal with the effect of a specific steady deformation. Rather, we consider the effect of a general axisymmetric deformation in order for the results to be applicable to other problems as well.

When the steady deformation is represented as

$$\zeta_s = \sum \alpha_j^{(s)} Y_j^0, \tag{4.17}$$

the effect of each mode is linearly additive. Thus, it is sufficient to consider a single mode of steady deformation

$$\zeta_s = \alpha_j^{(s)} Y_j^0. \tag{4.18}$$

For ζ_u and Φ_u, we assume

$$\zeta_u = \sum_{l,m}\alpha_{lm}(t)Y_l^m, \quad \Phi_u = \sum_{l,m}\beta_{lm}(t)r^{-(l+1)}Y_l^m. \tag{4.19}$$

Since we consider only the effect of steady deformation, we put $\Psi_s = 0$ and $\Psi_u = 0$. Then, from the kinematic condition, we have

$$\dot{\alpha}_{lm} = -(l+1)\beta_{lm} + \tfrac{1}{2}W\alpha_j^{(s)}\left[\sum_{p,q}\beta_{pq}\langle l,m|j,0|p,q\rangle\{l(l+1)+(p+1)(p+4)-j(j+1)\}\right] \quad (4.20)$$

where

$$\langle l,m|j,0|p,q\rangle = \int_0^{2\pi}\int_0^{\pi} Y_l^{m*}Y_j^0Y_p^q\ \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi, \quad (4.21)$$

and

$$\mathrm{L}^2Y_l^m = l(l+1)Y_l^m. \quad (4.22)$$

From the normal stress condition, we have

$$\dot{\beta}_{lm} = (l-1)(l+2)\alpha_{lm} + 3\kappa P_{g0}\alpha_{00}\delta_{l0} + W\alpha_j^{(s)}\left[\sum_{p,q}\{(p^3-3p)-2(j^2+j-1)\}\alpha_{pq}\langle l,m|j,0|p,q\rangle\right] + 3\kappa P_{g0}W\alpha_j^{(s)}\alpha_{00}\langle l,m|j,0|0,0\rangle, \quad (4.23)$$

where δ_{l0} is the Kronecker delta and the relationship

$$\dot{\beta}_{pq} = (p-1)(p+2)\alpha_{pq} + 3\kappa P_{g0}\alpha_{00}\delta_{p0} + O(W) \quad (4.24)$$

has been used in the rearrangement.

By using the properties of

$$\langle l,m|j,0|p,q\rangle = \int_0^{2\pi}\int_0^{\pi} Y_l^{m*}Y_j^0Y_p^q \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi,$$

which is non-zero only if (i) $m = q$ and (ii) $l - j \leqslant p \leqslant l + j$, and (iii) $l + p + j$ is even, we can rewrite the kinematic condition as

$$\dot{\alpha}_{lm} = -(l+1)\beta_{lm} + \tfrac{1}{2}\alpha_j^{(s)}W\left[\beta_{lm}\{2l^2+6l+4-j(j+1)\}\langle l,m|j,0|l,m\rangle + \cdots\right] \quad (4.25)$$

and the normal stress condition as

$$\dot{\beta}_{lm} = (l-1)(l+2)\alpha_{lm} + 3\kappa P_{g0}\alpha_{00}\delta_{l0} + \alpha_j^{(s)}W\left[\alpha_{lm}\{(l^3-3l)-2(j^2+j-1)\}\langle l,m|j,0|l,m\rangle + \cdots\right] + \alpha_j^{(s)}W\left[3\kappa P_{g0}\alpha_{00}\langle l,m|j,0|0,0\rangle\right]. \quad (4.26)$$

In the brackets of (4.25) and (4.26), terms such as $\beta_{l-j,m}$, $\beta_{l+j,m}$, $\alpha_{l-j,m}$, $\alpha_{l+j,m}$ $(j \neq 0)$ are not given explicitly because, as will be shown below, the effect of those terms on the oscillation frequency is at most $O(W^2)$. Thus, in the subsequent arguments, the effect of those terms is not considered.

We can first show by combining (4.25) and (4.26) that the dynamical equation for α_{lm} has the form (for simplicity, consider $l \neq 0$)

$$\ddot{\alpha}_{lm} + \left[G_0(l,m,j) + G_1(l,m,j)W\right]\alpha_{lm} = W\sum_{p\neq l} H(l,m,j,p)\alpha_{pm} + O(W^2), \tag{4.27}$$

where G_0, G_1, H are $O(1)$ coefficients for which we do not have to know the exact expressions at this point. Now we can show that the effect of the terms with α_{pm} ($p \neq l$) in the right-hand side of (4.27) is at most $O(W^2)$ by using the same argument as the one in Kang & Leal (1988) (see their equations (4.30)–(4.34)). But here we also touch on the idea very briefly. We substitute $\alpha_{lm} = \alpha_{lm0}e^{\lambda t}$, $\alpha_{pm} = \alpha_{pm0}e^{\lambda t}$ ($p \neq l$) into (4.27). Then we obtain a matrix which must be singular in order to have a non-trivial solution for α_{lm0} (after truncating the system of coupled equations at an arbitrarily large l). The matrix has the diagonal elements of $\lambda^2 + (G_0 + G_1W)$, and $O(W)$ or smaller off-diagonal elements resulting from the terms in the right-hand side of (4.27). In computation of the determinant of a matrix, any off-diagonal element must be multiplied by other off-diagonal elements at least once. Thus the contribution of the off-diagonal elements to the determinant is at most $O(W^2)$. Consequently the contribution of the terms on the right-hand side of (4.27) to the frequency is at most $O(W^2)$.

Now we can see from (4.26) that the volume mode oscillation ($l = 0$) behaves a little differently from the shape mode oscillation ($l \neq 0$). Thus we consider two cases separately.

(i) $l = 0$ (volume mode oscillation): when $l = 0$ and consequently $m = 0$, (4.25) and (4.26) can be combined to produce the dynamical equation for α_{00} which is valid to $O(W)$:

$$\ddot{\alpha}_{00} + (3\kappa P_{g0} - 2)\left[1 - W\frac{3}{(4\pi)^{1/2}}\alpha_j^{(s)}\left(\frac{\kappa P_{g0} - 2}{3\kappa P_{g0} - 2}\right)\delta_{j0}\right]\alpha_{00} = 0. \tag{4.28}$$

Therefore, the frequency of oscillation of the zeroth mode is modified as

$$\omega_{00}^2 = \omega_0^{*2}\left[1 - W\frac{3}{(4\pi)^{1/2}}\alpha_j^{(s)}\left(\frac{\kappa P_{g0} - 2}{3\kappa P_{g0} - 2}\right)\delta_{j0}\right] \tag{4.29}$$

where ω_0^* is the frequency of oscillation in the absence of an electric field and $\omega_0^{*2} = 3\kappa P_{g0} - 2$. The results (4.28) and (4.29) indicate that the zeroth mode (volume mode) is not influenced by the shape change (i.e. $j \geqslant 2$) but only by the volume change in steady deformation *up to the linear oscillation range*. However, in the range of higher-order nonlinear oscillation, it is known that there is an interaction between the volume and shape modes (Longuet-Higgins 1989*a,b*; Yang, Feng & Leal 1993). Now if we represent the steady volume change by using the Legendre polynomial as

$$\zeta_s = \alpha_j^{(s)}Y_j^0 = \tilde{\alpha}_j^{(s)}P_j(\cos\theta),$$

then

$$\omega_{00}^2 = \omega_0^{*2}\left[1 - W\frac{3}{(2j+1)^{1/2}}\tilde{\alpha}_j^{(s)}\left(\frac{\kappa P_{g0} - 2}{3\kappa P_{g0} - 2}\right)\delta_{j0}\right]. \tag{4.30}$$

(ii) $l \neq 0$ (shape mode): when $l \neq 0$, (4.25) and (4.26) can be combined as

$$\ddot{\alpha}_{lm} + (l-1)(l+1)(l+2)\left[1 - A_{02}^{(j)}(l,m)\alpha_j^{(s)}W\right]\alpha_{lm} = 0 \tag{4.31}$$

where

$$A_{02}^{(j)}(l,m) = \frac{(l-1)(l+2)\{l^2+3l+2-\frac{1}{2}j(j+1)\} - (l+1)\{l^3-3l-2(j^2+j-1)\}}{(l-1)(l+1)(l+2)}$$
$$\times \langle l,m|j,0|l,m\rangle. \tag{4.32}$$

In terms of the oscillation frequency, we have

$$\omega_{lm}^2 = \omega_l^{*2}\left[1 - A_{02}^{(j)}(l,m)\alpha_j^{(s)}W\right] \tag{4.33}$$

where

$$\omega_l^{*2} = (l-1)(l+1)(l+2).$$

The result (4.32) has a special meaning. By applying the addition theorem of spherical harmonics to (4.32), we can show that

$$\sum_{m=-l}^{l} A_{02}^{(j)}(l,m) = 0 \quad \text{if} \quad j \neq 0 \tag{4.34}$$

and

$$\sum_{m=-l}^{l} \omega_{lm}^2 = (2l+1)\omega_l^{*2} \quad \text{if} \quad j \neq 0. \tag{4.35}$$

The significance of result (4.35) is that the $(2l+1)$-fold degeneracy of oscillation frequency is split into different values depending on m due to the j-mode axisymmetric steady shape change. However, the average frequency is preserved. The same conclusion was obtained for drop oscillation in an alternating magnetic field (Cummings & Blackburn 1991).

Now it is worthwhile to consider some special cases.

(i) $j = 0$ (the effect of steady volume change): in this case, $A_{02}^{(0)}(l,m) = 3/(4\pi)^{1/2}$ and

$$\omega_{lm}^2 = \omega_l^{*2}\left[1 - 3\frac{\alpha_0^{(s)}}{(4\pi)^{1/2}}W\right]. \tag{4.36}$$

The above result can be verified quite easily by the following argument. When $\zeta_s = \alpha_0^{(s)}Y_0^0$, the deformed bubble has a spherical shape of changed radius. Thus the problem now is to compute the frequency of oscillation about the new spherical bubble of radius $a(1+W\zeta_s) = a[1+\alpha_0^{(s)}/(4\pi)^{1/2}W]$. Then if we take the new radius as the characteristic length scale instead of $l_c = a$, the dimensionless results are the same as before the volume change. Since the square of the dimensional frequency is inversely proportional to the third power of the characteristic length scale ($\hat{\omega}_l^2 = \omega_l^2/t_c^2 = \omega_{l0}^2(\gamma/\rho/l_c^3)$, where $\hat{\omega}_l$ and ω_l are the dimensional and the dimensionless frequencies of the lth mode oscillation, see the definitions of characteristic scales in the first paragraph of this section),

$$\hat{\omega}_{l,new} = \hat{\omega}_{l,old}\left[1 + \frac{\alpha_0^{(s)}}{(4\pi)^{1/2}}W\right]^{-3} = \hat{\omega}_{l,old}\left[1 - 3\frac{\alpha_0^{(s)}}{(4\pi)^{1/2}}W\right]O(W^2). \tag{4.37}$$

Therefore, we have the same result as in (4.36).

(ii) $j = 2$: this case is of special interest because the steady bubble shape in many situation includes a P_2-mode (or Y_2^0-mode) component as we have seen in the previous

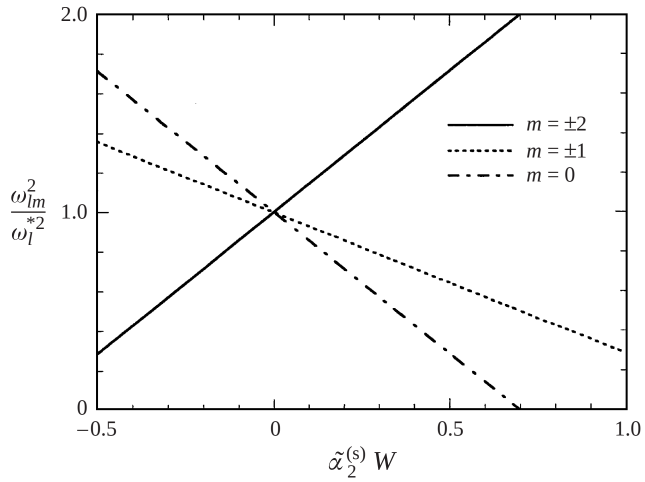

FIGURE 6. The frequency modification of Y_2^m-modes due to P_2-mode steady deformation (i.e. $R_s(\theta) = 1 + \tilde{\alpha}_2^{(s)} W P_2(\cos\theta)$).

section. When $j = 2$, (4.32) reduces to

$$A_{02}^{(2)}(l, m) = \frac{3(l^3 + l^2 + 2l + 4)}{(l-1)(l+1)(l+2)} \langle l, m|2, 0|l, m\rangle, \tag{4.38}$$

where

$$\langle l, m|2, 0|l, m\rangle = \left(\frac{5}{4\pi}\right)^{1/2} \left[\frac{3}{2}\left\{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)} + \frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right\} - \tfrac{1}{2}\right]. \tag{4.39}$$

In particular, for the axisymmetric oscillation ($m = 0$), (4.38) reduces to

$$A_{02}^{(2)}(l, 0) = \left(\frac{5}{4\pi}\right)^{1/2} \frac{3l(l^3 + l^2 + 2l + 4)}{(2l-1)(2l+3)(l-1)(l+2)}. \tag{4.40}$$

If the steady deformation is represented as

$$\zeta_s = \alpha_2^{(s)} Y_2^0 = \tilde{\alpha}_2^{(s)} P_2(\cos\theta), \tag{4.41}$$

the frequency of axisymmetric oscillation due to the P_2-mode steady shape change is given by

$$\frac{\omega_{l0}^2}{\omega_l^{*2}} = 1 - \tilde{A}_{02}^{(2)}(l, 0)\tilde{\alpha}_2^{(s)} W, \tag{4.42}$$

where

$$\tilde{A}_{02}^{(2)}(l, 0) = (4\pi/5)^{1/2} A_{02}^{(2)}(l, 0).$$

In figure 6, the frequency modification of $Y_2^m(\theta, \phi)$-modes due to the P_2-mode steady deformation ($R_s(\theta) = 1 + \tilde{\alpha}_2^{(s)} W P_2(\cos\theta)$) is shown. In this case,

$$\tilde{A}_{02}^{(2)}(2, 0) = \tfrac{10}{7}, \quad \tilde{A}_{02}^{(2)}(2, \pm 1) = \tfrac{5}{7}, \quad \tilde{A}_{02}^{(2)}(2, \pm 2) = -\tfrac{10}{7}. \tag{4.43}$$

As we can see in figure 6, the frequencies of Y_2^0-, $Y_2^{\pm 1}$-modes decrease while those of $Y_2^{\pm 2}$-modes increase as $\tilde{\alpha}_2^{(s)} W$ increases (the steady shape becomes more prolate). Another thing we should note is that the frequency of the Y_2^0-mode (axisymmetric

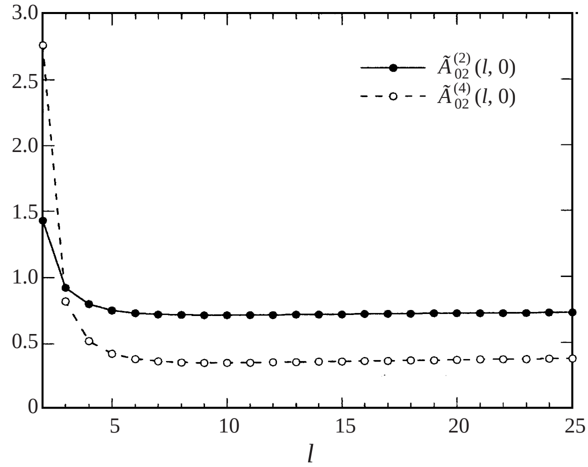

FIGURE 7. The effects of P_2- and P_4- mode steady deformation on the frequency modification of axisymmetric oscillation. The ordinate represents the values of the functions of l in the frequency modification equation given by $\omega_{l0}^2 = \omega_l^{*2}[1 - (\tilde{A}^{(2)}_{02}(l,0)\tilde{\alpha}_2^{(s)} + \tilde{A}^{(4)}_{02}(l,0)\tilde{\alpha}_4^{(s)})W]$ when $R_s(\theta) = 1 + [\tilde{\alpha}_2^{(s)}P_2(\cos\theta) + \tilde{\alpha}_4^{(s)}P_4(\cos\theta)]W$.

mode) can be larger than the natural frequency for the oblate steady shape ($\tilde{\alpha}_2^{(s)}W < 0$). The same behaviour, that the oblate deformation results in an increase of in the oscillation frequency, has also been observed for the drop problem experimentally (Trinh & Wang 1982) and theoretically (Kang 1993).

(iii) $j = 4$: equation (4.32) reduces to

$$A^{(4)}_{02}(l,m) = \frac{3l^3 - 4l^2 + 27l + 54}{(l-1)(l+1)(l+2)}\langle l,m|4,0|l,m\rangle. \tag{4.44}$$

Evaluation of $\langle l,m|4,0|l,m\rangle$ can be easily done by using the formula given in the Appendix. In particular, for the axisymmetric mode ($m = 0$) oscillation, we have

$$A^{(4)}_{02}(l,0) = \frac{9}{4}\left(\frac{9}{4\pi}\right)^{1/2}\frac{l(3l^3 - 4l^2 + 27l + 54)}{(2l-3)(2l-1)(2l+3)(2l+5)}. \tag{4.45}$$

If the steady deformation is represented as

$$\zeta_s = \alpha_4^{(s)}Y_4^0 = \tilde{\alpha}_4^{(s)}P_4(\cos\theta), \tag{4.46}$$

the frequency modification of axisymmetric oscillations due to the P_4-mode steady shape change is given by

$$\frac{\omega_{l0}^2}{\omega_l^{*2}} = 1 - \tilde{A}^{(4)}_{02}(l,0)\tilde{\alpha}_4^{(s)}W, \tag{4.47}$$

where

$$\tilde{A}^{(4)}_{02}(l,0) = (\tfrac{4}{9}\pi)^{1/2}A^{(4)}_{02}(l,0).$$

In figure 7, $\tilde{A}^{(2)}_{02}(l,0)$ and $\tilde{A}^{(4)}_{02}(l,0)$ are shown. As we can see from the figure with (4.42) and (4.47), if the steady deformation has the positive components of P_2- and P_4-modes, the frequencies of axisymmetric oscillation decrease due to the steady shape change.

4.2. *Special effects of electric fields*

As we have discussed already, the overall effect of an electric field on the bubble oscillation can be decomposed into two parts: (i) the indirect effect via the change in steady shape, and (ii) the direct effect of the electric field on the oscillation about a spherical shape. In this section, we consider the latter. Differently from the effects of steady deformation, the effect of an electric field on the oscillation is not linear because the electrical stress is a function of $\boldsymbol{EE}$. Thus we consider two specific electric fields: uniform and axisymmetric straining.

4.2.1. *The effect of a uniform electric field*

The uniform electric field about the spherical steady shape has the electric potential

$$\Psi_s = \left(r + \frac{1}{2r^2}\right) P_1(\cos\theta) = \left(\tfrac{4}{3}\pi\right)^{1/2} \left(r + \frac{1}{2r^2}\right) Y_1^0. \tag{4.48}$$

Thus we have

$$\Psi_s|_{r=1} = \tfrac{3}{2}(\tfrac{4}{3}\pi)^{1/2} Y_1^0 \tag{4.49}$$

and

$$\frac{\partial}{\partial r}(\nabla\Psi_s \cdot \nabla\Psi_s)|_{r=1} = 3\left(\tfrac{4}{5}\pi\right)^{1/2} Y_2^0 - 3(4\pi)^{1/2} Y_0^0. \tag{4.50}$$

For the unsteady electric field, we assume

$$\Psi_u = \sum_{l,m} \gamma_{lm}(t) r^{-(l+1)} Y_l^m. \tag{4.51}$$

By substituting equations (4.49) to (4.51) with $\zeta_s = 0$ into (4.13), (4.14), and (4.16), we obtain:

(i) kinematic condition:

$$\dot{\alpha}_{lm} = -(l+1)\beta_{lm}, \tag{4.52}$$

(ii) normal stress condition:

$$\begin{aligned}
&\tfrac{1}{2}W \Bigg[\tfrac{3}{2}\left(\tfrac{4}{3}\pi\right)^{1/2} \sum_{p,q} \gamma_{pq}\{l(l+1) - 2 - p(p+1)\}\langle l,m|1,0|p,q\rangle \\
&-3\left(\tfrac{4}{5}\pi\right)^{1/2} \sum_{p,q} \alpha_{pq}\langle l,m|2,0|p,q\rangle + 3(4\pi)^{1/2} \sum_{p,q} \alpha_{pq}\langle l,m|0,0|p,q\rangle \Bigg] \\
&-3\kappa P_{g0}\alpha_{00}\delta_{l0} + \dot{\beta}_{lm} = (l-1)(l+2)\alpha_{lm},
\end{aligned} \tag{4.53}$$

(iii) zero permittivity ratio condition:

$$(l+1)\gamma_{lm} = \tfrac{3}{4}\left(\tfrac{4}{3}\pi\right)^{1/2} \sum_{p,q} \alpha_{pq}\langle l,m|1,0|p,q\rangle\{l(l+1) + 2 - p(p+1)\}. \tag{4.54}$$

By using the properties of $\langle l,m|j,0|p,q\rangle$, and by combining the normal stress condition and the zero permittivity ratio condition, we have

$$\begin{aligned}
&-\tfrac{9}{4}W \left[\frac{(l-1)^2}{l} \tfrac{4}{3}\pi\langle l-1,m|1,0|l,m\rangle^2 + (l+2)\tfrac{4}{3}\pi\langle l+1,m|1,0|l,m\rangle^2 \right] \alpha_{lm} \\
&-\tfrac{3}{2}W(\tfrac{4}{5}\pi)^{1/2}\langle l,m|2,0|l,m\rangle\alpha_{lm} + \cdots + \tfrac{3}{2}W\alpha_{lm} \\
&-3\kappa P_{g0}\alpha_{00}\delta_{l0} + \dot{\beta}_{lm} = (l-1)(l+2)\alpha_{lm}.
\end{aligned} \tag{4.55}$$

In (4.55), the terms with $\alpha_{l-2,m}$ and $\alpha_{l+2,m}$ are not given explicitly because their effect is at most $O(W^2)$, shown by the same argument as the one below (4.26). Now, by combining (4.52) and (4.55), we get the dynamical equations for $\alpha_{lm}(t)$ as

(i) for $l = 0$

$$\ddot{\alpha}_{00} + (3\kappa P_{g0} - 2)\left[1 + \left(\frac{2}{3\kappa P_{g0} - 2}\right) A_{01}^{uni}(0,0)W\right]\alpha_{00} = 0, \tag{4.56}$$

(ii) for $l \geqslant 2$

$$\ddot{\alpha}_{lm} + (l-1)(l+1)(l+2)\left[1 - A_{01}^{uni}(l,m)W\right]\alpha_{lm} = 0, \tag{4.57}$$

where

$$A_{01}^{uni}(l,m) = -\frac{9}{4}\left[\frac{(l-1)}{l(l+2)}\tfrac{4}{3}\pi\langle l-1,m|1,0|l,m\rangle^2 + \frac{1}{l-1}\tfrac{4}{3}\pi\langle l+1,m|1,0|l,m\rangle^2\right]$$
$$+\frac{3}{2(l-1)(l+2)}\left[1 - \left(\tfrac{4}{5}\pi\right)^{1/2}\langle l,m|2,0|l,m\rangle\right] \tag{4.58}$$

and the expressions for $\langle l-1,m|1,0|l,m\rangle$, $\langle l+1,m|1,0|l,m\rangle$, $\langle l,m|2,0|l,m\rangle$ are given in the Appendix. Therefore, the direct effect of a uniform electric field on the frequency of oscillation about spherical shape can be represented as

(i) for $l = 0$:

$$\frac{\omega_{00}^2}{\omega_0^{*2}} = 1 + \left(\frac{2}{3\kappa P_{g0} - 2}\right) A_{01}^{uni}(0,0)W \tag{4.59}$$

where $\omega_l^{*2} = 3\kappa P_{g0} - 2$, and

(ii) for $l \geqslant 2$:

$$\frac{\omega_{lm}^2}{\omega_l^{*2}} = 1 - A_{01}^{uni}(l,m)W, \tag{4.60}$$

where $\omega_l^{*2} = (l-1)(l+1)(l+2)$.

From the above, we can see that

$$\sum_{l=-m}^{m} \omega_{lm}^2 \neq (2l+1)\omega_l^{*2} \tag{4.61}$$

because of $\langle l-1,m|1,0|l,m\rangle^2$ and $\langle l+1,m|1,0|l,m\rangle^2$. If we recall that the average oscillation frequency is preserved in the case of the steady deformation effect (equation (4.35)), we can see that the frequency modification characteristics are different in the case of the direct effect of an electric field.

In the case of axisymmetric oscillation, (4.60) reduces to

$$\frac{\omega_{l0}^2}{\omega_l^{*2}} = 1 - A_{01}^{uni}(l,0)W, \tag{4.62}$$

where

$$A_{01}^{uni}(l,0) = -\frac{9l(2l^3 + l^2 - 2l + 2)}{2(l-1)(l+2)(2l-1)(2l+1)(2l+3)}.$$

As we can see above, $A_{01}^{uni}(l,0)$ is negative for all mode numbers l. Thus, the frequency of axisymmetric oscillation increases due to the direct effect of uniform electric field. This point will be discussed later in detail.

4.2.2. *The effect of axisymmetric straining electric field*

When the undisturbed electric field is given in dimensionless form as $\boldsymbol{E}_\infty = \mathbf{G}\cdot\boldsymbol{x}$ with

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{4.63}$$

we have

$$\Psi_s = \left(\frac{r^2}{2} + \frac{1}{3r^3}\right) P_2(\cos\theta) = (\tfrac{4}{5}\pi)^{1/2} \left(\frac{r^2}{2} + \frac{1}{3r^3}\right) Y_2^0. \tag{4.64}$$

Now, for convenience we denote some important terms as

$$\Psi_s|_{r=1} = \gamma_2^{(s)} Y_2^0, \tag{4.65}$$

and

$$\frac{\partial}{\partial r}(\nabla\Psi_s \cdot \nabla\Psi_s)|_{r=1} = \eta_0^{(s)} Y_0^0 + \eta_2^{(s)} Y_2^0 + \eta_4^{(s)} Y_4^0, \tag{4.66}$$

where

$$\gamma_2^{(s)} = \tfrac{5}{6}\left(\tfrac{4}{5}\pi\right)^{1/2}, \quad \eta_0^{(s)} = -\tfrac{5}{3}(4\pi)^{1/2}, \quad \eta_2^{(s)} = -\tfrac{25}{21}\left(\tfrac{4}{5}\pi\right)^{1/2}, \quad \eta_4^{(s)} = \tfrac{20}{7}\left(\tfrac{4}{9}\pi\right)^{1/2}.$$

The kinematic condition is again given by (4.52), but the normal stress condition is given by

$$\begin{aligned} &\tfrac{1}{2}W\left[\gamma_2^{(s)} \sum_{p,q} \gamma_{pq}\{l(l+1) - 6 - p(p+1)\}\langle l,m|2,0|p,q\rangle\right] \\ &\quad -\tfrac{1}{2}W\left[\sum_{p,q} \alpha_{pq}\{\eta_0^{(s)}\langle l,m|0,0|p,q\rangle + \eta_2^{(s)}\langle l,m|2,0|p,q\rangle + \eta_4^{(s)}\langle l,m|4,0|p,q\rangle\}\right] \\ &\quad -3\kappa P_{g0}\alpha_{00}\delta_{l0} + \dot{\beta}_{lm} = (l-1)(l+2)\alpha_{lm}. \end{aligned} \tag{4.67}$$

The zero permittivity ratio condition becomes

$$\begin{aligned} &(l+1)\gamma_{lm} - 6\gamma_2^{(s)} \sum_{p,q} \alpha_{pq}\langle l,m|2,0|p,q\rangle \\ &\quad = \tfrac{1}{2}\gamma_2^{(s)}\left[\sum_{p,q} \alpha_{pq}\langle l,m|2,0|p,q\rangle\{l(l+1) - 6 - p(p+1)\}\right]. \end{aligned} \tag{4.68}$$

Then by using the properties of $\langle l,m|2,0|p,q\rangle$, we have the following result from (4.68):

$$\begin{aligned} \gamma_{lm} = \gamma_2^{(s)}&\left[2\alpha_{l-2,m}\langle l,m|2,0|l-2,m\rangle + \left(\frac{3}{l+1}\right)\alpha_{lm}\langle l,m|2,0|l,m\rangle\right. \\ &\left. -\left(\frac{2l}{l+1}\right)\alpha_{l+2,m}\langle l,m|2,0|l+2,m\rangle\right] \end{aligned} \tag{4.69}$$

By substituting (4.69) into (4.67), we have

$$\dot{\beta}_{lm} = (l-1)(l+2)\alpha_{lm} + 3\kappa P_{g0}\alpha_{00}\delta_{l0} + W(\gamma_2^{(s)})^2 \left[\frac{4(l-2)^2}{l-1} \langle l,m|2,0|l-2,m\rangle^2 + \frac{9}{l+1}\langle l,m|2,0|l,m\rangle^2 + 4(l+3)\langle l,m|2,0|l+2,m\rangle^2 \right] \alpha_{lm} + \tfrac{1}{2} W[\eta_0^{(s)}\langle l,m|0,0|l,m\rangle + \eta_2^{(s)}\langle l,m|2,0|l,m\rangle + \eta_4^{(s)}\langle l,m|4,0|l,m\rangle]\alpha_{lm} + \cdots. \quad (4.70)$$

By combining (4.70) and the kinematic condition, we have the dynamical equation for $\alpha_{lm}(t)$:

(i) for $l = 0$:

$$\ddot{\alpha}_{00} + (3\kappa P_{g0} - 2)\left[1 + \left(\frac{2}{3\kappa P_{g0} - 2}\right) A_{01}^{str}(0,0)W\right]\alpha_{00} = 0, \quad (4.71)$$

(ii) for $l \geqslant 2$:

$$\ddot{\alpha}_{lm} + (l-1)(l+1)(l+2)\left[1 - A_{01}^{str}(l,m)W\right]\alpha_{lm} = 0, \quad (4.72)$$

where

$$A_{01}^{str}(l,m) = -\frac{25}{36(l-1)(l+2)}\left[\frac{4(l-2)^2}{l-1}\tfrac{4}{5}\pi\langle l,m|2,0|l-2,m\rangle^2 + \frac{9}{l+1}\left(\tfrac{4}{5}\pi\right)\langle l,m|2,0|l,m\rangle^2 + 4(l+3)\left(\tfrac{4}{5}\pi\right)\langle l,m|2,0|l+2,m\rangle^2\right] - \frac{1}{2(l-1)(l+2)}[-\tfrac{5}{3}(4\pi)^{1/2}\langle l,m|0,0|l,m\rangle - \tfrac{25}{21}\left(\tfrac{4}{5}\pi\right)^{1/2}\langle l,m|2,0|l,m\rangle + \tfrac{20}{7}\left(\tfrac{4}{9}\pi\right)^{1/2}\langle l,m|4,0|l,m\rangle]. \quad (4.73)$$

In (4.73), the coefficients $\gamma_2^{(s)}$, $\eta_0^{(s)}$, $\eta_2^{(s)}$, $\eta_4^{(s)}$ defined below (4.66) are substituted. Therefore, the direct effect of a straining electric field is

(i) for $l = 0$:

$$\frac{\omega_{00}^2}{\omega_0^{*2}} = 1 + \left(\frac{2}{3\kappa P_{g0} - 2}\right) A_{01}^{str}(0,0)W, \quad (4.74)$$

where $\omega_l^{*2} = 3\kappa P_{g0} - 2$, and

(ii) for $l \geqslant 2$:

$$\frac{\omega_{lm}^2}{\omega_l^{*2}} = 1 - A_{01}^{str}(l,m)W, \quad (4.75)$$

where $\omega_l^{*2} = (l-1)(l+1)(l+2)$. In particular when $m = 0$ (axisymmetric oscillation),

$$A_{01}^{str}(l,0) = -\frac{25(8l^8 + 20l^7 - 22l^6 - 37l^5 + 53l^4 - 4l^3 - 72l^2 + 15l - 9)}{2(l-1)(l+2)(2l-3)(2l-1)^2(2l+1)(2l+3)^2(2l+5)} \quad (4.76)$$

which is negative for all $l \geqslant 2$. Therefore, as in the case of a uniform electric field, the straining electric field has the direct effect of increasing the frequency of axisymmetric oscillation for all the mode numbers $l \geqslant 2$. This point will be discussed later in detail.

4.3. *Overall effect of a uniform electric field*

Recall that the electric field affects the oscillation frequency in two ways: (i) via the change in steady shape, and (ii) by the direct effect on the oscillation of a bubble about spherical shape. We now put the results of the two effects from previous sections together to see the overall effect.

In the case of a uniform electric field, we have the steady deformation (from (3.21))

$$\zeta_s = \alpha_0^{(s)} Y_0^0 + \alpha_2^{(s)} Y_2^0, \tag{4.77}$$

with

$$\alpha_0^{(s)} = -\frac{3}{4(3\kappa P_{g0} - 2)}(4\pi)^{1/2}, \quad \alpha_2^{(s)} = \tfrac{3}{16}\left(\tfrac{4}{5}\pi\right)^{1/2}.$$

The above result is combined with the results in the previous sections to get the overall frequency change. For the volume mode oscillation ($l = 0$), the results in (4.29) and (4.62) are combined to get

$$\frac{\omega_{00}^2}{\omega_0^{*2}} = 1 - 3\frac{\alpha_0^{(s)}}{(4\pi)^{1/2}}\left(\frac{\kappa P_{g0} - 2}{3\kappa P_{g0} - 2}\right) W, \tag{4.78}$$

where $A_{01}^{uni}(0,0) = 0$ has been used and $\omega_0^{*2} = 3\kappa P_{g0} - 2$. By substituting the result (4.77), we have

$$\frac{\omega_{00}^2}{\omega_0^{*2}} = 1 + \frac{9}{4(3\kappa P_{g0} - 2)}\left(\frac{\kappa P_{g0} - 2}{3\kappa P_{g0} - 2}\right) W. \tag{4.79}$$

For the shape mode oscillation ($l \geqslant 2$), the overall effect of a uniform electric field can be written as

$$\frac{\omega_{lm}^2}{\omega_l^{*2}} = 1 - \left[A_{01}^{uni}(l,m) + A_{02}^{(0)}(l,m)\alpha_0^{(s)} + A_{02}^{(2)}(l,m)\alpha_2^{(s)}\right] W, \tag{4.80}$$

where $A_{01}^{uni}(l,m)$ is given in (4.58), $A_{02}^{(0)}(l,m)$ in (4.36), and $A_{02}^{(2)}(l,m)$ in (4.38). In (4.80), $A_{01}^{uni}(l,m)W$ represents the frequency decrease due to the direct effect of the uniform electric field on the oscillation about the undeformed spherical shape. Other terms $A_{02}^{(0)}(l,m)\alpha_0^{(s)}W$ and $A_{02}^{(2)}(l,m)\alpha_2^{(s)}W$ represent the frequency decrease of free oscillation due to the deformed steady shape given in the form of (4.77). If the gas inside the bubble is incompressible, $\alpha_0^{(s)} = 0$, and (4.80) reduces to

$$\frac{\omega_{lm}^2}{\omega_l^{*2}} = 1 - \left[A_{01}^{uni}(l,m) + A_{02}^{(2)}(l,m)\alpha_2^{(s)}\right] W. \tag{4.81}$$

In particular, for the axisymmetric oscillation ($m = 0$), we have

$$\frac{\omega_{l0}^2}{\omega_l^{*2}} = 1 - \left[A_{01}^{uni}(l,0) + A_{02}^{(2)}(l,0)\alpha_2^{(s)}\right] W, \tag{4.82}$$

where

$$A_{01}^{uni}(l,0) = -\frac{9l(2l^3 + l^2 - 2l + 2)}{2(l-1)(l+2)(2l-1)(2l+1)(2l+3)}, \tag{4.83}$$

$$A_{02}^{(2)}(l,0)\alpha_2^{(s)} = \frac{9l(l^3 + l^2 + 2l + 4)}{16(2l-1)(2l+3)(l-1)(l+2)}. \tag{4.84}$$

For axisymmetric modes of oscillation of a bubble in a uniform electric field, the two independent effects and the combined effect are shown in figure 8. As we can see in the figure, the oscillation frequency increases due to the direct effect of the uniform electric field on oscillations about spherical shape. Note that $A_{01}^{uni}(l,0) < 0$ for all l and $A_{01}^{uni}(l,0) \to 0$ as $l \to \infty$. On the other hand, the deformed steady shape has the effect of frequency decrease. Note that $A_{02}^{(2)}(l,0)\alpha_2^{(s)} > 0$ for all l and $A_{02}^{(2)}(l,0)\alpha_2^{(s)} \to \frac{9}{64}$ as $l \to \infty$. These opposite tendencies make an interesting result for the overall effect of an electric field. The frequencies of axisymmetric modes increase for lower mode numbers ($l \leqslant 6$), while they decrease for higher mode numbers ($l \geqslant 7$) as shown in figure 8.

The above two effects may be explained as follows. In the uniform electric field, a bubble is deformed into a prolate shape and the arclength between the two poles is increased compared with the spherical shape. In the case of unforced free axisymmetric oscillation, the frequency decreases as the arclength increases, which may explain the effect of deformed shape on the frequency change. On the other hand, the direct effect of an electric field requires another explanation. In the case of a bubble with the permittivity ratio $q = \varepsilon_{in}/\varepsilon \to 0$, the electric field is always tangential to the bubble surface ($E_n = \boldsymbol{n}\cdot\boldsymbol{E} = 0$). Thus the electrical normal stress is given by

$$T_{nn}^{(e)} = -\tfrac{1}{2}\varepsilon E_t^2, \qquad (4.85)$$

which means the electrical field exerts a suppressive force on the surface. Now, if some part of the surface protrudes from the spherical surface, near the protruded point the electric field becomes stronger due to the concentration of electric field. Thus, the bubble is experiencing a relatively stronger suppressive force near the protruded point. This suppressive force adds to the returning action by the surface tension. If some part is indented, the electric field becomes weaker near that point. Thus the suppressive force is weaker than the case of a spherical shape, and again this weaker suppressive force adds to the returning action. Since the electric field has an effect of increased surface tension, the oscillation frequency is expected to be increased. Since we have not assumed any specific electric field in the discussion of the direct effect, it may be stated that any type of electric field has an effect of increasing the oscillation frequency about a spherical shape as long as the liquid is a perfect dielectric.

It is worthwhile to compare the above results with those of a conducting drop in a uniform electric field, which are available in Kang (1993). The steady deformation into a prolate shape of a conducting drop has the effect of decreasing the oscillation frequency as in the case of bubble. However, differently from the case of a bubble, the oscillation frequency of a conducting drop about spherical shape decreases in the uniform electric field. The effect of the deformed steady drop shape can also be explained by the extended arclength because the drop is also deformed into the prolate shape. The direct effect on the oscillation about a spherical shape can be explained as following. The electric field about the conducting drop has only the normal component at the drop surface and the electrical normal stress is given by

$$T_{nn}^{e} = \tfrac{1}{2}\varepsilon E_n^2, \qquad (4.86)$$

which means that the electric field exerts a pulling force on the surface (recall that the electric field exerts a suppressive force in the bubble case). If there is a protruded part of the drop surface, the normal stress becomes larger due to the concentrated electric field about the protruded part and the electrical normal stress pulls out the surface more strongly than the spherical case. Thus once there is a protruded part, the electrical stress retards the action of surface tension to return to the spherical shape. If there is an indented part, the pulling force due to the electric field is weaker than the spherical case and it again retards the surface tension action. Thus, the electric field has an effect that is equivalent to reduced surface tension. Consequently, the frequency of oscillation is decreased due to the electric field in the case of a conducting drop.

4.4. *Overall effect of a straining electric field*

In the case of a straining electric field, we have the steady deformation (from (3.26))

$$\zeta_s = \alpha_0^{(s)} Y_0^0 + \alpha_2^{(s)} Y_2^0 + \alpha_4^{(s)} Y_4^0 \tag{4.87}$$

with

$$\alpha_0^{(s)} = -\frac{5}{12(3\kappa P_{g0}-2)}(4\pi)^{1/2}, \quad \alpha_2^{(s)} = -\frac{25}{336}\left(\frac{4\pi}{5}\right)^{1/2}, \quad \alpha_4^{(s)} = \frac{5}{126}\left(\frac{4\pi}{9}\right)^{1/2}.$$

Let us first consider the case of volume oscillation ($l = 0$). The results in (4.30) and (4.71) can be combined to get

$$\begin{aligned}\frac{\omega_{00}^2}{\omega_0^{*2}} &= 1 + \frac{2}{3\kappa P_{g0}-2} A_{01}^{str}(0,0)W - 3\frac{\alpha_0^{(s)}}{(4\pi)^{1/2}}\left(\frac{\kappa P_{g0}-2}{3\kappa P_{g0}-2}\right)W \\ &= 1 + \frac{5}{12(3\kappa P_{g0}-2)}\left[2 + 3\left(\frac{\kappa P_{g0}-2}{3\kappa P_{g0}-2}\right)\right]W,\end{aligned} \tag{4.88}$$

where $A_{01}^{str}(0,0) = 5/12$ has been used and $\omega_0^{*2} = 3\kappa P_{g0} - 2$.

When $l \geqslant 2$ (shape mode oscillation), the overall effect of a straining electric field can be written as

$$\frac{\omega_{lm}^2}{\omega_l^{*2}} = 1 - \left[A_{01}^{str}(l,m) + A_{02}^{(0)}(l,m)\alpha_0^{(s)} + A_{02}^{(2)}(l,m)\alpha_2^{(s)} + A_{02}^{(4)}(l,m)\alpha_4^{(s)}\right]W, \tag{4.89}$$

where $A_{01}^{str}(l,m)$ is given in (4.73), $A_{02}^{(0)}(l,m)$ in (4.36), $A_{02}^{(2)}(l,m)$ in (4.38), and $A_{02}^{(4)}(l,m)$ in (4.44). If the gas inside the bubble is incompressible, $\alpha_0^{(s)} = 0$, and (4.89) reduces to

$$\frac{\omega_{lm}^2}{\omega_l^{*2}} = 1 - \left[A_{01}^{str}(l,m) + A_{02}^{(2)}(l,m)\alpha_2^{(s)} + A_{02}^{(4)}(l,m)\alpha_4^{(s)}\right]W. \tag{4.90}$$

In particular for the axisymmetric oscillation, we have

$$\frac{\omega_{l0}^2}{\omega_l^{*2}} = 1 - \left[A_{01}^{str}(l,0) + A_{02}^{(2)}(l,0)\alpha_2^{(s)} + A_{02}^{(4)}(l,0)\alpha_4^{(s)}\right]W \tag{4.91}$$

with

$$A_{01}^{str}(l,0) = -\frac{25(8l^8 + 20l^7 - 22l^6 - 37l^5 + 53l^4 - 4l^3 - 72l^2 + 15l - 9}{2(l-1)(l+2)(2l-3)(2l-1)^2(2l+1)(2l+3)^2(2l+5)} \tag{4.92}$$

and

$$A_{02}^{(2)}(l,0)\alpha_2^{(s)} = -\frac{25l(l^3 + l^2 + 2l + 4)}{112(2l-1)(2l+3)(l-1)(l+2)}, \tag{4.93}$$

$$A_{02}^{(4)}(l,0)\alpha_4^{(s)} = \frac{5l(3l^3 - 4l^2 + 27l + 54)}{56(2l-3)(2l-1)(2l+3)(2l+5)}. \tag{4.94}$$

The results for the axisymmetric bubble oscillation in a straining electric field are shown in figure 9.

As discussed in the subsection for the uniform electric field, a straining electric field also increases the frequency of oscillation about a spherical shape. In the case of the deformed steady shape effect, the positive components of both P_2- and P_4-modes have the effect of decreasing oscillation frequency. However, in the case of the straining electric field, the steady deformation has a negative component for the P_2-mode and a positive component for the P_4-mode.

4.5. *Weak viscosity effect*

As mentioned earlier, the effects of the electric field and the viscosity are independent of each other to $O(W)$ or $O(S)$. Thus for the present work, it is sufficient to consider the viscosity effect on the oscillation of a bubble about a spherical shape. The result for the viscous effect was first obtained using the dissipation theory by Lamb (1932), but here we touch on the problem very briefly according to the formulation in this section. For oscillation about a spherical shape with a weak viscosity effect, we have the kinematic condition

$$\frac{\partial \zeta_s}{\partial t} = \frac{\partial \phi_u}{\partial r} \tag{4.95}$$

and the normal stress condition

$$\frac{\partial \Phi_u}{\partial t} + 2S\frac{\partial^2 \Phi_u}{\partial r^2} - \tilde{p}_v - 3\kappa\langle\zeta_u\rangle P_{g0} = -2\zeta_u + L^2(\zeta_u). \tag{4.96}$$

By using the results in Prosperetti (1977) or Kang & Leal (1988) for $\tilde{p}_v$, we can show from the kinematic condition

$$\dot{\alpha}_{lm} = -(l+1)\beta_{lm} \tag{4.97}$$

and from the normal stress condition

$$\dot{\beta}_{lm} + 2S(2l+1)(l+2)\beta_{lm} - 3\kappa P_{g0}\alpha_{00}\delta_{l0} = (l-1)(l+2)\alpha_{lm}. \tag{4.98}$$

By combining (4.97) and (4.98), we have

(i) $l = 0$:

$$\ddot{\alpha}_{00} + 4S\dot{\alpha}_{00} + (3\kappa P_{g0} - 2)\alpha_{00} = 0, \tag{4.99}$$

(ii) $l \neq 0$:

$$\ddot{\alpha}_{lm} + 2S(2l+1)(l+2)\dot{\alpha}_{lm} + (l-1)(l+1)(l+2)\alpha_{lm} = 0. \tag{4.100}$$

Therefore the effect of viscosity on the oscillation frequency is $O(S^2)$ and given by

(i) $l = 0$:

$$\frac{\omega_{00}^2}{\omega_0^{*2}} = 1 - \frac{4}{(3\kappa P_{g0} - 2)}S^2 \tag{4.101}$$

where $\omega_0^{*2} = 3\kappa P_{g0} - 2$,

(ii) $l \geqslant 2$:

$$\frac{\omega_{lm}^2}{\omega_l^{*2}} = 1 - D_0(l)S^2 \tag{4.102}$$

where $\omega_l^{*2} = (l-1)(l+1)(l+2)$ and

$$D_0(l) = \frac{(2l+1)^2(l+2)}{(l-1)(l+1)}.$$

5. Concluding remarks

In the present work, a three-dimensional analysis is performed to investigate the effects of an electric field on the steady deformation and the small-amplitude oscillation of a bubble in a dielectric liquid. As a first attempt, the simplest case of a bubble in a perfect dielectric liquid is considered. However, in order to deal with a general class of electric fields, an arbitrary electric field near the bubble is approximated by the sum of a uniform field and a linear field. The electric field inside and outside the bubble is obtained first by using the leaky dielectric model of Melcher & Taylor (1969) under the assumption that the surface current is negligible. Then the problem of a bubble in a perfect dielectric liquid is considered as a special case.

For steady deformation of a bubble in an arbitrary electric field, a general formula is obtained. It is found that, to the first-order effect of an electric field, the steady bubble shape can be in general represented by a linear combination of a finite number of spherical harmonics Y_l^m, where $0 \leqslant l \leqslant 4$ and $|m| \leqslant l$.

In an electric field, a bubble does not oscillate about the spherical shape but it oscillates about the deformed steady shape. Thus, we need to consider also the indirect effect of an electric field on the frequency change via the deformed steady shape in addition to the direct effect on the oscillation about a spherical shape. To the first order, the two effects are of the same order and mutually independent. The overall frequency change can thus be decomposed into two parts: (i) free oscillation about the *deformed steady shape*, and (ii) oscillation about the *undeformed spherical shape* due to the direct effect of the electric field. The deformed steady shape has an effect that splits the $(2l+1)$-fold degeneracy of oscillation frequency of Y_l^m modes, in the case of oscillation about the spherical shape, into the different frequencies that depend on m. However, when the average is taken over the $2l+1$ different m values, the average frequency is preserved. On the other hand, the splitting of oscillation frequencies due to the direct effect does not preserve the average frequency.

The frequency modification of axisymmetric modes has been studied in more detail and it has been found that the direct and the indirect effects show opposite tendencies with the increase of electric field. For axisymmetric modes of an incompressible bubble, deforming the steady shape into a prolate spheroid has an effect that decreases the oscillation frequency due to the extended pole-to-pole arclength of the prolate spheroid. However, it has been found that the electric field increases the frequency of oscillation about a spherical shape. For a bubble, the electric field exerts a suppressive force on the surface, which becomes relatively stronger at protruding parts of the surface and relatively weaker at indented parts. According to this mechanism, the electric field strengthens the action of surface tension, and consequently the frequency of oscillation about the undeformed spherical shape increases in electric fields.

As in the bubble case, the overall frequency change of a conducting drop can be decomposed into the same two parts: A conducting drop has a prolate steady shape in a uniform electric field and the frequency of axisymmetrc oscillation decreases due to this steady shape change as in the case of bubble. However, the drop shows different characteristics for the direct effect of electric field. The electric field outside the drop exerts a pulling force on the surface, and the protruded part of surface experiences a relatively stronger pulling force and this weakens the action of surface tension. In this way, the electric field reduces the frequency of oscillation about the spherical shape. Thus, in the case of conducting drop, both the direct and indirect effects decrease the frequency of axisymmetric oscillation. Indeed, it was found that the frequency of axisymmetric oscillation decreases in a uniform electric field for all

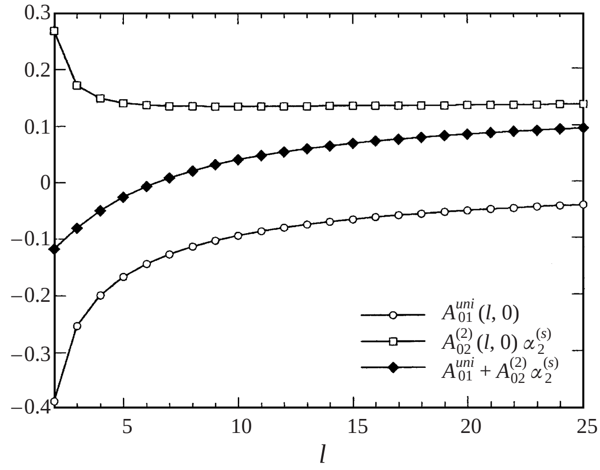

FIGURE 8. The effects of a uniform electric field on the frequency modification of axisymmetric oscillation. The ordinate represents the values of the functions of l in the frequency modification equation given by $\omega_{l0}^2 = \omega_l^{*2}[1 - (A_{01}^{uni} + A_{02}^{(2)}\alpha_2^{(s)})W]$. The term $A_{01}^{uni}W$ denotes the direct effect and $A_{02}^{(2)}\alpha_2^{(s)}W$ denotes the indirect effect due to the steady shape change.

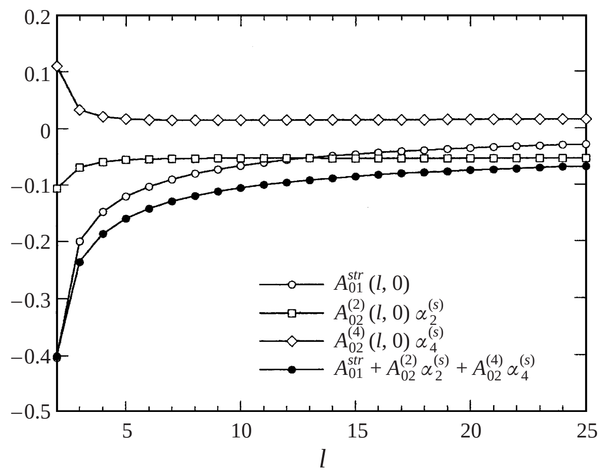

FIGURE 9. The effects of a straining electric field on the frequency modification of axisymmetric oscillation. The ordinate represents the values of the functions of l in the frequency modification equation given by $\omega_{l0}^2 = \omega_l^{*2}[1 - (A_{01}^{str} + A_{02}^{(2)}\alpha_2^{(s)} + A_{02}^{(4)}\alpha_4^{(s)})W]$. The term $A_{01}^{str}W$ denotes the direct effect and $(A_{02}^{(2)}\alpha_2^{(s)} + A_{02}^{(4)}\alpha_4^{(s)})W$ denotes the indirect effect due to the steady shape change.

mode numbers l in the case of conducting drop (Kang 1993). On the other hand, in the case of bubble, the indirect effect due to deformed steady shape reduces the frequency of axisymmetric oscillation, while the direct effect on the oscillation about spherical shape increases the oscillation frequency. Thus, the overall frequency change is a function of the mode number and it has been observed from figure 8 that the frequency decreases for lower mode numbers ($l \leqslant 6$) while it increases for higher mode numbers ($l \geqslant 7$).

Now it is appropriate to mention the experimental realizability of the results obtained from this study. Various types of non-uniform electric fields can be obtained by changing the shapes and configurations of electrodes (e.g. Jones & Bliss 1977; Pohl 1978; Bellini *et al.* 1997). The approximate electric field near the bubble can be obtained by applying (2.1) to the electric field solution for the overall system of electrodes. The oscillation frequency of a specific mode can be found by selectively exciting the mode by using a weak time-periodic electric field (Bellini *et al.* 1997) or an acoustic field (Trinh & Wang 1982).

Finally it should be mentioned that, in the present study, the simplest case of a perfect dielectric liquid has been discussed and charge transport has not been considered. For more realistic applications, however, charge transport should also be considered. In our future works, the analysis will be improved by including its effects such as electrokinetic effects (Saville 1997).

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Appendix

The formulas for $\langle l,m|j,0|p,q\rangle$ in some special cases are as follows:

$$(4\pi)^{1/2}\langle l,m|0,0|l,m\rangle = 1, \tag{A 1}$$

$$\left(\tfrac{4}{3}\pi\right)^{1/2}\langle l,m|1,0|l-1,m\rangle = A(l-1,m), \tag{A 2}$$

$$\left(\tfrac{4}{3}\pi\right)^{1/2}\langle l,m|1,0|l+1,m\rangle = A(l,m), \tag{A 3}$$

$$\left(\tfrac{4}{5}\pi\right)^{1/2}\langle l,m|2,0|l-2,m\rangle = \tfrac{3}{2}A(l-1,m)A(l-2,m), \tag{A 4}$$

$$\left(\tfrac{4}{5}\pi\right)^{1/2}\langle l,m|2,0|l,m\rangle = \tfrac{3}{2}\left[A(l,m)^2 + A(l-1,m)^2\right] - \tfrac{1}{2}, \tag{A 5}$$

$$\left(\tfrac{4}{5}\pi\right)^{1/2}\langle l,m|2,0|l+2,m\rangle = \tfrac{3}{2}A(l,m)A(l+1,m), \tag{A 6}$$

$$\begin{aligned}\left(\tfrac{4}{9}\pi\right)^{1/2}\langle l,m|4,0|l,m\rangle = {}& \tfrac{35}{8}\left[A(l,m)^2\left\{A(l+1,m)^2 + A(l,m)^2 + A(l-1,m)^2\right\}\right.\\ &\left. + A(l-1,m)^2\left\{A(l,m)^2 + A(l-1,m)^2 + A(l-2,m)^2\right\}\right]\\ &-\tfrac{15}{4}\left[A(l,m)^2 + A(l-1,m)^2\right] + \tfrac{3}{8},\end{aligned} \tag{A 7}$$

where

$$A(l,m) = \left[\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}\right]^{1/2}.$$

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